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Modeling and Analysis of a Permanently Magnetized Sphere's Motion Facilitated by Field Manipulation

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***Abstract*— A model describing the motion of a magnetized sphere in a fluid medium subject to an applied field is developed. The inspiration for this system is rooted in technologies and procedures being explored in the biomedical community. In these systems, the accepted approximation of Stokes' flow about the sphere is discussed and limitations to this are identified with drag force relations accounting for wall effects presented. For this system, the sphere is approximated as a point dipole and the resulting electromagnetic force model is presented. For motion of the sphere using static electromagnets, a reduced coil set technique is proposed. The technique consists of utilizing the minimum number of coils for generation of the desired force on the sphere. This approach is in contrast to the existing solutions which use the full coil set for parallel and anti-parallel motion and single out a solution for the underdetermined system. A technique for determining the proper coil combination is developed from examining the definiteness of the geometric field functions. A coil array configuration consisting of a four coil assembly is investigated to facilitate planar motion of a magnetized spherical particle in a fluid medium. For this configuration, the reduced coil set consists of only two adjacent coils being active at any given time. The exact inverse current solution for the minimized coil set is derived and presented. The system dynamic model is formulated and represented in a nonlinear state space system.**

I. Introduction

In recent decades many new technologies have been proposed and investigated in the biomedical sciences. Several studies have suggested the use of magnetic nano-particles *in vivo* to the human body. The benefits of such methods are believed to be invaluable due to the non-invasive capabilities of anticipated procedures. Envisioned applications of nano-particles include, but are not limited to, cell labeling, magnetic separation, targeted drug delivery, hyperthermic cell treatment, and MRI contrast enhancement [1]. Significant study has been performed regarding these concepts and is available in references [2], [3], [4], [5]. A second novel application of a magnetically reactive device being introduced to the internal workings of a living body and being manipulated by an applied *ex vivo* magnetic field is magnetic implant guidance. The practice of this procedure on the human brain is referred to as stereotactic neurosurgery and a significant effort has been put forth on this technique [6], [7]. Similarly, a process utilizing nano sized particles and applied external magnetic fields for intraocular retinal repair is being explored as well [8]. Finally, the fascinating and emerging science of nano-scaled machines has offered the potential employment of microdevices in medical procedures. To reduce device complexity, an external magnetic propulsion technique is envisioned [9], [10].

The common attribute present in all of the aforementioned systems is the use of an applied magnetic field. This field is utilized to exert a desired force and/or torque on an object susceptible to such fields. Upon that realization, one may attempt to first recognize the analogy and similarities to the classic magnetic levitation problem. Indeed a parallel may be observed that manipulation of a particle or micro-device along a unidirectional trajectory is comparable to the well-known technique of suspending or manipulating objects by either the application of alternating fields or utilizing a direct field to maintain the static position of the target object [11], [12]. However, in the aforementioned biomedical processes, the ability to either sustain a static position in a moving fluid medium possessing varying components of velocity in multiple coordinate directions, or propel the target object(s) along a predetermined spatial trajectory elevates this problem to one beyond classical levitation. Achieving this type of dynamics requires application of forces with components in multiple coordinate directions. The resulting fields, field gradients, and the means by which to generate them are far more complex than that of the levitation problem, which often requires force generation only along a single direction. A system such as this presents an interesting control problem for which a suitable descriptive model is necessary before a sufficient treatment and analysis may occur.

The technique of moving an implant or micro-device by fields from static electromagnets is not a new concept [6], [7], [13]. Indeed a substantial amount of work has been performed using a cubic arrangement of coils to facilitate three dimensional motion. A recent technique as described in [13] utilizes six active coils to impart a predetermined force and torque on a target object. The studied approach powers all coils simultaneously with the three opposite the primary attractive coils being operated with a lower current of reverse polarity. This would result in an increase in the field gradients at the location of the target and in effect raise the magnetic force on the object. While this technique has been explored, calculation of the required currents is quite tedious. The device operates in an open loop control scheme and employs minimization techniques to numerically solve for the six coil currents. The difficulty in the current solution for this system rests with the fact that it is underdetermined (i.e., there are more equations than unknowns). For the scenario of moving a spherical object where the orientation is unconstrained, a solution of six unknowns (the currents) from three equations

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is needed. Hence, many solutions exist, requiring the use of optimization methods to single out a minimal current set. This paper explores the technique of reducing the number of coils to only those which are necessary to generate the force. This results in a *determined* system, allowing for an exact calculation of the current set. The analysis will be limited to the less complicated problem of moving a sphere two dimensionally in a fluid medium using static electromagnets. The system consists of four electromagnets, of which only two are used at any time to provide planar motion to the particle. The pair of active coils which will be utilized is determined by the force requirement on the sphere. The dynamic model of the sphere motion along with the electromagnetic model of force generation is developed in Section II. The exact inverse current solution for the reduced set is derived in Section III. Finally, closed loop control aspects are discussed and the model to be used for control design is presented in section IV.

II. Dynamic Model Development

The equation of motion describing the movement of the target object may be devised through a Newtonian formulation given here as

$$m\ddot{\vec{S}}(t) = \vec{F}_{Bias} + \vec{F}_{Mag} + \vec{F}_{Drag}, \tag{1}$$

where $\vec{F}_{Bias}$ denotes any bias force, $\vec{F}_{Mag}$ denotes the applied magnetic force, $\vec{F}_{Drag}$ denotes a fluid drag force from the motion of the object relative to the fluid medium it resides in, m denotes the mass of the object, and $\ddot{\vec{S}}(t)$ denotes the vector acceleration of the object in an inertial reference frame. The corresponding trajectory $\vec{S}(t)$ describing the object's motion may be ascertained by solving the differential equation (1). For the system consisting of a spherical target residing in a fluid medium, the first term on the right hand side of (1) may be defined to represent the resulting force of gravity on the sphere and the buoyant force of the fluid medium on the sphere

$$\vec{F}_{Bias} = (\rho_{sph} - \rho_{fl})V_{sph}\vec{g}, \tag{2}$$

where ρ_{sph} denotes the density of the sphere, ρ_{fl} denotes the density of the fluid medium in which the sphere resides, V_{sph} denotes the volume of the sphere, and $\vec{g}$ denotes the acceleration of the sphere due to gravitational attraction. The second term on the right hand side of (1) may be expressed by approximating the sphere as a magnetic dipole. This approximation for small objects is widely accepted and has seen some experimental correlation [2], [8]. When a magnetic field is applied, a resulting force and torque act on the dipole. In this case, assuming that the fluid medium has a permeability close to that of free space, the force and torque on the sphere are

$$\vec{F}_{Mag}(\mathbf{H}) = \frac{4\pi\mu_0 r^3}{3}\Big(\mathbf{M}\cdot\nabla\Big)\mathbf{H} \tag{3a}$$

$$\vec{\tau}_{Mag}(\mathbf{H}) = \frac{4\pi\mu_0 r^3}{3}(\mathbf{M}\times\mathbf{H}), \tag{3b}$$

where $\mathbf{M}$ is the magnetization of the body, μ_0 is the permeability of free space (valued at $4\pi\times10^{-7}\mathrm{T}\cdot\mathrm{m/A}$), r is the radius of the sphere, and $\mathbf{H}$ is the applied magnetic field strength at the location of the sphere [14], [15]. Note that boldface indicates a vector field. In the case of the particle residing in a medium where the relative permeability $\mu_{fl} \gg 1$, then $\mu_0 \rightarrow \mu_0\mu_{fl}$. Recognizing that rotational resistance of the spherical object will be small in most fluid media, it is apparent from (3b) that the body will have a tendency to align with the applied field $\mathbf{H}$. It may be noted that in some applications, particularly those involving an elongated magnetic object as suggested in [13], the alignment of the object relative to the applied force $\vec{F}_{Mag}$ may be considered as an additional constraint on the target. In this study, however, this requirement is disregarded due to the symmetry of the object.

At this point in the model formulation, the type of material utilized for the spherical target must be addressed, as this affects the magnetization $\mathbf{M}$ achieved when the field $\mathbf{H}$ is applied. Primarily two types of materials are proposed for use, paramagnetic or ferromagnetic. The paramagnetic class of material is characterized by a linear relationship between the resulting magnetization and the internal magnetic field of the object. This linear constant is referred to as the magnetic *susceptibility* and is typically denoted as χ_m. The resulting magnetic force on the paramagnetic sphere is [14]

$$\vec{F}_{Mag,PM}(\mathbf{H}) = \frac{2\pi\mu_0 r^3(\chi_m)}{(\chi_m+3)}\nabla\big(\mathbf{H}\cdot\mathbf{H}\big).$$

The second class of material, ferromagnetic, is characterized by a nonlinear relationship between the applied magnetic field and the magnetization of the object. More specifically, the $M-H$ curve is sigmoidal in shape indicative of these materials reaching a magnetic saturation. Additionally, the sigmoidal $M-H$ curve may also possess a region of hysteresis about the origin depending on the classification of the ferromagnetic material as "hard" or "soft". Hard ferromagnetic materials are typified as those in which the $M-H$ hysteresis loop is large and upon removal of the applied H, the sample is permanently magnetized (e.g. a permanent magnet). This is often referred to as magnetic *remanence*. In contrast, soft ferromagnetic materials have a small hysteresis loop and very little residual magnetization upon removal of any applied magnetic field (e.g. transformer iron) [16]. This work will address only the ferromagnetic material type. For the ferromagnetic sphere residing in a medium with permeability close to that of free space, the resultant magnetic force for a given amount of magnetization is

$$\vec{F}_{Mag,FM}(\mathbf{H}) = \frac{4\pi\mu_0 r^3}{3}\left(\frac{M}{H}\mathbf{H}\cdot\nabla\right)\mathbf{H}, \tag{4}$$

where M is the magnitude of magnetization and H is the magnitude of the applied field $\mathbf{H}$ at the sphere's location [14]. For the permanently magnetized sphere, M may be treated as a constant. In the Cartesian coordinate system, taking $\mathbf{H} = H_X\hat{i} + H_Y\hat{j} + H_Z\hat{k}$ where $\hat{i}$, $\hat{j}$, and $\hat{k}$ are unit vectors in the respective coordinate directions, the magnitude of the applied field is given by $H = |\mathbf{H}| = \sqrt{H_X^2 + H_Y^2 + H_Z^2}$.

For the final force on the right hand side of (1), it has been readily accepted in the literature [1], [2], [3], [4], [5], [8], [10] that the flow about a spherical particle or object in these applications corresponds to a low Reynolds' number condition (i.e. $Re \ll 1$) and that the drag force may be formulated by applying the Stokes' flow unbounded free stream drag force relation

$$\vec{F}_{Drag} = 6\pi r\mu(\dot{\vec{S}}_{fl} - \dot{\vec{S}}), \tag{5}$$

where μ represents the viscosity of the fluid medium, $\dot{\vec{S}}_{fl}$ represents the free stream velocity and the quantity $(\dot{\vec{S}}_{fl} - \dot{\vec{S}})$ denotes the velocity of the moving object relative to the free stream velocity. Indeed there is experimental evidence that, at least for the cases where particles are manipulated in a stagnant or slowly moving medium, this approximation correlates with the observed phenomena [2], [8]. It must be acknowledged, however, that flow scenarios may exist in which the Stokes approximation (5) may loose some accuracy. For instance, the Reynolds' number for flow over a sphere is

$$Re = \frac{2\rho_{fl} S_{fl} r}{\mu},$$

where ρ_{fl} is the density of the fluid. The extreme smallness of the particle may still allow for a calculation of $Re \ll 1$ even with a significant magnitude of flow velocity around the sphere. If the object is of a larger size (e.g. a microdevice or catheter guide), the result of $Re \ll 1$ may no longer hold, pushing the drag coefficient into a nonlinear function of Re. Additionally, the relative size of the sphere to that of the channel in which it is moving and the geometry of that channel will have an impact on the calculation of the drag coefficient. As an example, consider the system of a sphere moving along the central axis of a cylindrical duct or a square duct. For the cylindrical duct, the drag force on the sphere is given as

$$F_{Drag,cyl} = 6\pi\mu U r\Big(1 + k\frac{r}{R_0}\Big), \tag{6}$$

where U is the velocity of the sphere, R_0 is the radius of the cylindrical duct, and k is a correction coefficient valued at $k = 2.10444$ [17]. For the square duct, the drag force on the sphere is given by

$$F_{Drag,sqr} = 6\pi\mu U r\Big(1 + k\frac{r}{l}\Big), \tag{7}$$

where l is the half width of the square duct and $k = 1.903266$ [17]. Likewise, the wall effects on an object in close proximity to the channel surface have been neglected. These will influence the drag force and often impart lift and moment components on the sphere. Depending on the flow boundary conditions, often the most convenient way to evaluate these effects is through numerical simulation [18], [19]. Additionally, it is well known that fluids in living bodies do not behave in a "Newtonian" fashion. Additional forces may be present, i.e. collisions with other small bodies. Depending on the size of the target object, these dynamics may need to be accounted for. Finally, the condition of turbulent flow in the medium is neglected. Turbulent flow of the fluid medium will lead to significant deviation from the Stokes' approximation. The wall and turbulent flow effects will be ignored for the purposes of this study and left for analysis at a later time. For now, equations (2), (4), and (5) may be used in (1) to solve for the motion of the sphere. A component which remains to be addressed is the method to generate the required field $\mathbf{H}$ in (4).

The technique utilized in this study for generation of the magnetic field on the sphere is that of an air core electromagnet. It will be assumed that the required motion of the target and hence the required field changes will be slow enough that the magnetic *quasistatic* approximation may be used to ignore any transient effects (e.g. eddy currents, etc.). In this system, stationary electromagnets will be employed. Because of this, multiple coils must be used to facilitate parallel and anti-parallel forces along multiple coordinate directions. This is due to the fact that when the sphere is unrestrained in the rotational sense and allowed to freely align with the field, a single electromagnet will act in an attractive capacity alone. For three dimensional motion an obvious configuration is to make use of six coils arranged in a cubic fashion similar to that of the magnetic implant guidance system by *Stereotaxis, Inc.* [6], [7], [13]. The resulting field from a closed path current density conforms to a solution of Maxwell's field equations which are readily available in [14], [15]. Application of these equations and subsequent reduction of this set to the law of Biot-Savart yields a relation expressing the field $\mathbf{H}$ of the P-th individual coil as a linear function of the current applied to the coil

$$\mathbf{H}_P = j_P[G_X\hat{i} + G_Y\hat{j} + G_Z\hat{k}] = j_P\mathbf{G}_P(X,Y,Z), \tag{8}$$

where j_P is the applied current, $\hat{i}$, $\hat{j}$, and $\hat{k}$ are unit vectors in the respective coordinate directions, and $\mathbf{G}_p$ is a nonlinear vector field multiplier, which is a function of the location of the sphere relative to the coil and the geometry of the coil itself. The combined field from the employment of Q electromagnets may be ascertained through superposition of the individual coil fields

$$\mathbf{H} = [J]^T \begin{bmatrix} \mathbf{G}_1 \\ \vdots \\ \mathbf{G}_Q \end{bmatrix} = [J]^T[\mathbf{G}], \tag{9}$$

where $[J]$ is a $Q \times 1$ column vector of the applied currents ($[J] \in \mathbb{R}^Q$) and $\mathbf{G}_P$ is the geometry and position dependent vector field from the P-th coil evaluated in an inertial coordinate system. Using (9) the following quantity in (4) is derived

$$\begin{aligned}\Big(\frac{M}{H}\mathbf{H}\cdot\nabla\Big)\mathbf{H} &= \\ \frac{M}{H}[J]^T\Big([\mathbf{k}_{FM,X}]\hat{i} &+ [\mathbf{k}_{FM,Y}]\hat{j} + [\mathbf{k}_{FM,Z}]\hat{k}\Big)[J] \\ &= \frac{M}{H}[J]^T[\vec{\mathbf{k}}_{FM}][J], \end{aligned} \tag{10}$$

where

$$\mathbf{k}_{FM,X} = [G_X]\left[\frac{\partial G_X}{\partial X}\right]^T + [G_Y]\left[\frac{\partial G_X}{\partial Y}\right]^T + [G_Z]\left[\frac{\partial G_X}{\partial Z}\right]^T,$$

$$\mathbf{k}_{FM,Y} = [G_X]\left[\frac{\partial G_Y}{\partial X}\right]^T + [G_Y]\left[\frac{\partial G_Y}{\partial Y}\right]^T + [G_Z]\left[\frac{\partial G_Y}{\partial Z}\right]^T,$$

$$\mathbf{k}_{FM,Z} = [G_X]\left[\frac{\partial G_Z}{\partial X}\right]^T + [G_Y]\left[\frac{\partial G_Z}{\partial Y}\right]^T + [G_Z]\left[\frac{\partial G_Z}{\partial Z}\right]^T,$$

$$H = \sqrt{[J]^T\Big[[G_X][G_X]^T + [G_Y][G_Y]^T + [G_Z][G_Z]^T\Big][J]},$$

$$[G_S] = \begin{bmatrix} G_{S,1} \\ \vdots \\ G_{S,Q} \end{bmatrix}, \quad (11a)$$

$$\left[\frac{\partial G_S}{\partial S}\right]^T = \left[\frac{\partial G_{S,1}}{\partial S}, \cdots, \frac{\partial G_{S,Q}}{\partial S}\right]. \quad (11b)$$

Note that in equations (11a) and (11b), $S = X, Y, Z$. All field components and partial derivatives of the field components are evaluated in a global inertial coordinate system. Substituting (2), (4), (5), and (10) into (1) yields the equation of motion

$$m\ddot{\vec{S}} = (\rho_s - \rho_{fl})V_s\vec{g} + \frac{4\pi\mu_0 r^3}{3}\frac{M}{H}[J]^T[\vec{\mathbf{k}}_{FM}][J] + 6\pi r\mu(\dot{\vec{S}}_{fl} - \dot{\vec{S}}). \quad (12)$$

Observe that (12) relates a given column vector of input current to the resultant motion of the sphere. Because this system utilizes electromagnet coils, the inductive effects of the electromagnet array must be taken into account. Hence, the equation of motion must be augmented by a suitable description of the electrical dynamics for a complete description of the system. Approximating each coil as an R-L circuit (a resistor in series with an inductor) results in the following relationship between the input voltage and current flow of the electromagnets

$$v_m = R_m j_m + \sum_{n=1}^{Q} L_{m,n}\frac{dj_n}{dt}, \quad (13)$$

where Q is the number of coils, $m = 1, 2, \ldots, Q$, $n = 1, 2, \ldots, Q$, v_m denotes the input voltage of coil m, R_m denotes the resistance of coil m, j_m denotes the current in coil m, and dj_n/dt denotes the time rate of change of the current in coil n. Additionally, $L_{m,n}$ signifies the mutual magnetic inductance of coil n on coil m or the self inductance of coil m when $n = m$ [14]. It should be noted that while the self inductance is always positively valued, the sign of the mutual inductance is dependent on the orientation of coil m to n. Observing that (13) is a linear system, it may be rewritten in state space form

$$[\dot{J}] = -[L^{-1}][R][J] + [L^{-1}][V] \quad (14)$$

where $[V]$ and $[J]$ are $Q \times 1$ column vectors of the voltages and currents in Q coils, $[R]$ is a $Q \times Q$ diagonal matrix of the coil resistances, and $[L]$ is a $Q \times Q$ symmetric matrix of inductance with the diagonal elements being the self inductance terms and the off diagonal elements being the mutual inductance terms. Note that $[L]$ is symmetric because $L_{n,m} = L_{m,n}$ and that $[L]$ is invertible [14]. Equation (12) is used in concert with (14) to provide a full description of the sphere subject to Stokes' flow and a magnetic field generated by Q discrete electromagnets with input voltages $[V]$.

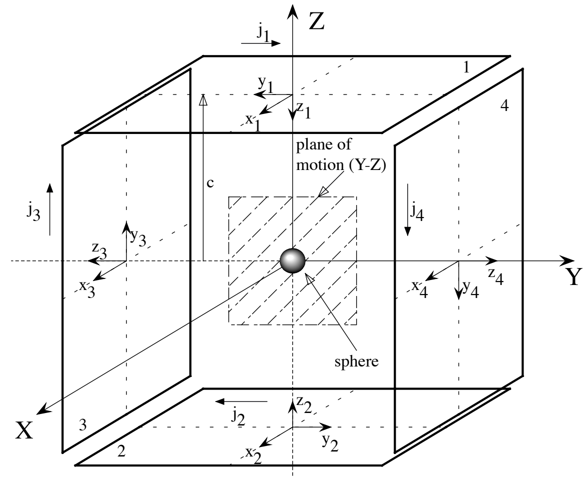

Fig. 1. Schematic of a four coil arrangement to provide forces for planar motion of a sphere. The individual coil coordinates indicate direction of positive current polarity (right hand rule).

III. Inverse Current Solution for Constrained 2-D Motion with a Reduced Coil Set

In an effort to further explore a system using stationary coils to manipulate the position of a magnetized sphere in a fluid medium, this study will examine the two dimensional planar motion case with the sphere constrained to the $Y-Z$ plane (see Fig. 1). As previously mentioned, a cubic arrangement of coils can facilitate three dimensional motion of the target sphere [6], [7], [13]. Similarly, a set of four coils arranged on the edges of a plane with the central axes of the coils lying in the plane will facilitate forced motion in that plane. A schematic of this type of system is shown in Fig. 1. The inverse current solution entails determining the current trajectories $[J]$ which would cause the sphere to move along a desired motion trajectory $\vec{S}$. Indeed, the inverse current solution may be considered as a more thorough examination of (4), acknowledging in this case that $\vec{F}_{Mag}$ is the force required to "offset" the inertial, bias, and drag forces incurred by the sphere moving along $\vec{S}$. It is obvious from (4) that $\vec{F}_{Mag,FM}$ is a function linear in $[J]$ but is nonlinear in the components of $\mathbf{H}$, so a direct algebraic solution is not feasible. With this noted, it is evident that in the configuration where the sphere is constrained to a central plane and when the coil configuration to provide both parallel and anti-parallel motion is employed, the system is underdetermined. In other words, a solution in four variables (the coil currents) is to be ascertained from just two equations (the vector components of (4) in the Y and Z directions). Due to the fact

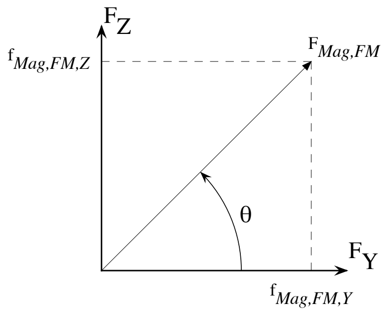

Fig. 2. Graphical representation of the applied planar magnetic force.

that electromagnets will act in an attractive capacity alone, one may reduce the number of utilized coils to only those which are necessary for the required force. In the scenario of planar motion utilizing the system in Fig. 1, this entails reducing the active coils to that of "corner" combinations (i.e. coils 1&4, 1&3, 2&4, and 2&3). This will reduce the number of solution variables from four to two and make the system determined. For example, suppose that there is a known required magnetic force $\vec{F}_{Mag,FM}$ in the $Y-Z$ plane. This force may be represented in a cylindrical coordinate system as $\vec{F}_{Mag,FM} = f_{Mag,FM}[\cos(\theta)\hat{j} + \sin(\theta)\hat{k}]$ where $f_{Mag,FM}$ is the force magnitude and θ gives the angular direction of the force measured from the (global) Y axis with the counter clockwise direction taken as positive (see Fig. 2). Recognizing that the coils will only attract the sphere, for a required force in the direction of $\theta = 45°$, the necessary coils to achieve this will be numbers 1 and 4 (see Fig. 1) with the actual current amplitudes and polarity dependent on the location of the sphere. In other words, the coil combination for a force pointing into a given quadrant will be the coils which bound that quadrant. One may postulate that the corner combinations of 1&4, 1&3, 2&3, and 2&4 correspond to force directions of $0° \leq \theta \leq 90°$, $90° \leq \theta \leq 180°$, $180° \leq \theta \leq 270°$, and $270° \leq \theta \leq 360°$, respectively. However, this quadrant assignment to pairs of coils is approximate due to the nonlinear nature of the attractive magnetic fields.

The components of the required force $\vec{F}_{Mag,FM}$ may be expressed as a function of one another

$$f_{Mag,Z} = \tan(\theta) f_{Mag,Y}, \tag{15}$$

where once again θ is the angular direction of the required force measured from the Y axis with the counter clockwise direction taken as positive. As can be seen from (4) and (12), the Y and Z components of magnetic force for the ferromagnetic sphere are

$$f_{Mag,Y} = \frac{4\pi\mu_0 r^3}{3}\frac{M}{H}[J]^T[\mathbf{k}_{FM,Y}][J], \tag{16a}$$

$$f_{Mag,Z} = \frac{4\pi\mu_0 r^3}{3}\frac{M}{H}[J]^T[\mathbf{k}_{FM,Z}][J]. \tag{16b}$$

Substituting (16a) and (16b) into (15), rearranging, and then simplifying yields

$$[J]^T[\mathbf{\Omega}][J] = 0, \tag{17}$$

where

$$[\mathbf{\Omega}] = \begin{cases} [\mathbf{k}_{FM,Y}], \text{ for } \theta = \frac{\pi}{2} \pm n\pi, \; n = 0, 1, 2\ldots; \\ [\mathbf{k}_{FM,Z}] - \tan(\theta)[\mathbf{k}_{FM,Y}], \text{ otherwise.} \end{cases}$$

The quantity $[J]$ is a 2×1 vector of the currents in the reduced coil set and the vector fields $[\mathbf{k}_{FM,Y}]$ and $[\mathbf{k}_{FM,Z}]$ are 2×2 matrices containing the $\mathbf{G}$ and $\partial\mathbf{G}/\partial S$ terms of the reduced set in accordance with (11a) and (11b). The matrices $[\mathbf{k}_{FM,Y}]$ and $[\mathbf{k}_{FM,Z}]$ are given by the quantities in (10), however G_S and $\partial G_S/\partial S$ are now 2×1 matrices with the first element corresponding to coil 1 or 2 and the second element corresponding to coil 3 or 4, depending on the coil combination being examined. Once again note that all field components and partial derivatives of the field components are evaluated in the global inertial coordinate system. Equation (17) corresponds to a quadratic surface in the J (current) space, centered about the origin. For a current solution other than $J = [0 \; 0]^T$ to exist, $[\mathbf{\Omega}]$ must be indefinite or semi-definite [20]. In other words, for a nontrivial solution to exist $[\mathbf{\Omega}]$ cannot be positive or negative definite. Several techniques are available to ascertain the definiteness of $[\mathbf{\Omega}]$. One method involves examining the signs of the principle minors of $[\mathbf{\Omega}]$. If the principle minors are not exclusively < 0 or > 0, then $[\mathbf{\Omega}]$ is semi-definite or indefinite [20]. Alternatively, for the 2×2 matrix $[\mathbf{\Omega}]$ one may examine the product of eigenvalues. If the eigenvalue product is ≤ 0, then $[\mathbf{\Omega}]$ is semi-definite or indefinite. To facilitate the inverse current solution, it is necessary to require the eigenvalues to be real. This may be achieved by forcing symmetry through the relation $[J]^T[\mathbf{\Omega}][J] = [J]^T[(\mathbf{\Omega} + \mathbf{\Omega}^T)/2][J] = [J]^T[\mathbf{\Delta}][J]$. Recognizing that the 2×2 matrix $[\mathbf{\Omega}]$ of the reduced coil set will yield two eigenvalues, the eigenvalues of $[\mathbf{\Delta}]$ will be denoted as λ_1 and λ_2. For a sphere located in the $Y-Z$ plane, a force may be effected on the sphere by any coil combination with an eigenvalue product of $\lambda_1\lambda_2 \leq 0$. As an example, the sign of the eigenvalue products for coil combination 1&4 was examined over a range of $-45° \leq \theta \leq 135°$ for three different locations in the $Y-Z$ plane. The system analyzed consists of rectangular current loops with a major dimension of 1 unit, a minor dimension of 0.5 units and an offset "c" from the global origin (see Fig. 1) of 1.5 units. The effective force angle range evaluated at the $Y-Z$ locations of $(0.5, 0.5)$, (0.0), and $(-0.5, -0.5)$ are $-30.51° \leq \theta \leq 120.51°$, $-2.43° \leq \theta \leq 82.43°$, and $10.89° \leq \theta \leq 79.11°$, respectively. These angular ranges indicate the directions of force that coil combination 1&4 is capable of achieving at the given sphere locations. As alluded to earlier, the range of force directions a particular coil combination may impart on the sphere is dependent on the nonlinear field expressions which are functions of the sphere's position as well as coil geometry. The effective range of force direction decreases as the sphere moves away from the electromagnet pair.

To solve for the required currents to achieve the desired (planar) force $\vec{F}_{Mag,FM}$, the orthonormal basis $[\widetilde{\mathbf{Q}}]$ composed of the normalized eigenvectors of $[\mathbf{\Delta}]$ may be used to form the orthogonal transformation $[J] = [\widetilde{\mathbf{Q}}][\widetilde{J}]$.

Application of this to $[J]^T[\mathbf{\Delta}][J] = 0$ results now in the diagonal quadratic $[\widetilde{J}]^T[\widetilde{\mathbf{D}}][\widetilde{J}] = 0$ where $[\widetilde{\mathbf{D}}]$ is a diagonal matrix of the eigenvalues of $[\mathbf{\Delta}]$. In the $\widetilde{J}$ space, two line loci may now be formed from the diagonal quadratic

$$\tilde{j}_1^2\lambda_1 + \tilde{j}_2^2\lambda_2 = 0 \begin{cases} \tilde{j}_2 = \pm\tilde{j}_1\sqrt{\dfrac{-\lambda_1}{\lambda_2}}; \ \lambda_2 \neq 0, & (18a) \\ \tilde{j}_1 = \pm\tilde{j}_2\sqrt{\dfrac{-\lambda_2}{\lambda_1}}; \ \lambda_1 \neq 0. & (18b) \end{cases}$$

Note that (18) actually provides two separate formulations for the line loci with a caveat on the existence of a non-zero eigenvalue. Utilizing (18), the $[\widetilde{J}]$ space current vector may now be rewritten as

$$[\widetilde{J}] = \begin{cases} \tilde{j}_1 \begin{bmatrix} 1 \\ \pm\sqrt{\frac{-\lambda_1}{\lambda_2}} \end{bmatrix} = \tilde{j}_1[\xi_1^{\pm}]; \ \lambda_2 \neq 0, & (19a) \\ \tilde{j}_2 \begin{bmatrix} \pm\sqrt{\frac{-\lambda_2}{\lambda_1}} \\ 1 \end{bmatrix} = \tilde{j}_2[\xi_2^{\pm}]; \ \lambda_1 \neq 0. & (19b) \end{cases}$$

Applying the orthogonal transformation to (16a) yields

$$f_{Mag,Y} = \frac{4\pi\mu_0 r^3}{3}\frac{M}{\widetilde{H}}[\widetilde{J}]^T[\widetilde{\mathbf{Q}}]^T[\mathbf{k}_{FM,Y}][\widetilde{\mathbf{Q}}][\widetilde{J}], \quad (20)$$

where

$$\widetilde{H} = \sqrt{[\widetilde{J}]^T[\widetilde{\mathbf{Q}}]^T[G][\widetilde{\mathbf{Q}}][\widetilde{J}]},$$

$$[G] = G_X G_X^T + G_Y G_Y^T + G_Z G_Z^T.$$

Substitution of (19a) into (20), simplifying, and utilizing (18a) results in the $[\widetilde{J}]$ solution of

$$\tilde{j}_1 = \pm\frac{3f_{Mag,Y}}{4\pi\mu_0 r^3 M}\frac{\sqrt{[\xi_1^{\pm}]^T[\widetilde{\mathbf{Q}}]^T[G][\widetilde{\mathbf{Q}}][\xi_1^{\pm}]}}{[\xi_1^{\pm}]^T[\widetilde{\mathbf{Q}}]^T[\mathbf{k}_{FM,Y}][\widetilde{\mathbf{Q}}][\xi_1^{\pm}]}, \quad (21a)$$

$$\tilde{j}_2 = \pm\tilde{j}_1\sqrt{\frac{-\lambda_1}{\lambda_2}}. \quad (21b)$$

Note that (21) must still conform to the caveat of $\lambda_2 \neq 0$. For the condition that $\lambda_2 = 0$, a similar process may be employed utilizing (18b) and (19b) to provide the solution

$$\tilde{j}_1 = \pm\tilde{j}_2\sqrt{\frac{-\lambda_2}{\lambda_1}}, \quad (22a)$$

$$\tilde{j}_2 = \pm\frac{3f_{Mag,Y}}{4\pi\mu_0 r^3 M}\frac{\sqrt{[\xi_2^{\pm}]^T[\widetilde{\mathbf{Q}}]^T[G][\widetilde{\mathbf{Q}}][\xi_2^{\pm}]}}{[\xi_2^{\pm}]^T[\widetilde{\mathbf{Q}}]^T[\mathbf{k}_{FM,Y}][\widetilde{\mathbf{Q}}][\xi_2^{\pm}]}, \quad (22b)$$

where $\lambda_1 \neq 0$. Additionally, solutions may be formulated using (16b). This will result in forms similar to (21) and (22) with $f_{Mag,Y} \rightarrow f_{Mag,Z}$ and $[\mathbf{k}_{FM,Y}] \rightarrow [\mathbf{k}_{FM,Z}]$. To select the proper signs for the solution coordinates, the correct groupings on loci intersections must be observed. These are available in Table I, where in the solution value $\tilde{j}_\delta^{\alpha\beta}$ ($\alpha, \beta = \pm$, $\delta = 1, 2$), $\alpha = \pm$ denotes the sign just to the right of the $=$ symbol in (21) or (22) and $\beta = \pm$ denotes the sign used for $[\xi_\delta^\beta]$, as defined in (19). It can be seen from (21), (22), and Table I that for the planar motion reduced set, there are four different solution coordinates. The coordinate system $\widetilde{J}$ is a transformed system and therefore has no physical meaning. To complete the calculation of the current solution, the $\widetilde{J}$ coordinate solutions must be transformed back to the J space via $[J] = [\widetilde{\mathbf{Q}}][\widetilde{J}]$. A plot of the intersecting loci indicating the solution coordinates in the J space for the previously mentioned coil configuration, a sphere with a unit parameter multiplier $(4\pi\mu_0 r^3 M/3 = 1)$ located at the global origin, and a required magnetic force of $\vec{F}_{Mag} = 1\hat{j} + 1\hat{k}$ is shown in Fig. 3. As can be seen, the four solution coordinates may be ascertained by the intersection of the component loci of equation(s) (16) or the intersection of the line loci for equation (17) with just one of the component loci of (16). Indeed it is the latter technique which yields the results of (21) and (22). The calculated solution coordinates using (21) for solutions J_a, J_b, J_c, and J_d are $(11.208, 11.208)$, $(-11.208, -11.208)$, $(-25.519, 25.519)$, and $(25.519, -25.519)$, respectively. As can be seen, they correlate with the loci intersections.

TABLE I
CORRECT SIGNS FOR $\widetilde{J}$ SPACE SOLUTION COORDINATES

solution set	sign
$\widetilde{J}_a$	$(\tilde{j}_1^{++}, \tilde{j}_2^{++})$
$\widetilde{J}_b$	$(\tilde{j}_1^{-+}, \tilde{j}_2^{-+})$
$\widetilde{J}_c$	$(\tilde{j}_1^{+-}, \tilde{j}_2^{--})$
$\widetilde{J}_d$	$(\tilde{j}_1^{--}, \tilde{j}_2^{+-})$

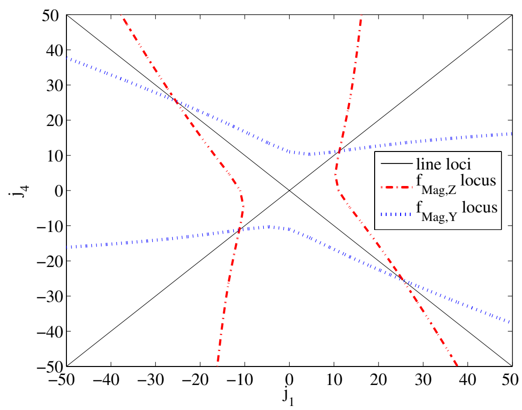

Fig. 3. Solution loci for the 1&4 coil combination in the J space.

The inverse current solution presented in this work was derived for the ferromagnetic material type. A similar derivation may be used for the paramagnetic material and will give the same number of solutions, four. However, the solution is of a simpler nature because the shapes of the component loci for a given force are analytically known. In this case the current solution coordinates correspond to intersecting conic sections. Extending this line of thought would result in the inverse current solution for the paramagnetic material being manipulated by a cubic arrangement of coils being the intersection of quadric surfaces.

IV. State Space Model and Closed Loop Control Considerations

Implementation of closed loop control on this type of system is challenging and has not been explored in literature yet. Even with the reduced set and the exact calculation of the required currents, it has been shown in this paper that for the two dimensional case there will be four possible solutions. For the three dimensional case utilizing six stationary coils, the reduced set would number three and the number of solution coordinates would be eight. Because of this, closed loop control employing the reduced set will be challenging as well. Even with that said, the benefits of being able to exactly calculate the possible solutions are obvious when compared with using optimization methods to search out solutions for an underdetermined system. Feedback control will be further complicated for this system due to the inductive effects of the coils. For a given (slow) trajectory, the current solutions to follow that path may be calculated from (21) or (22). The voltage solutions would then need to be evaluated using (13). The method of using a reduced coil set would inherently involve "switching" coils on and off. Inductive effects associated with this type of technique must be accounted and compensated for. One advantage of this system is that for the electrical dynamics, full state feedback is available. Therefore it is conceivable that the switching effects may be taken into account. In the case of large coils with significant inductance parameters, full state feedback of current may prove to be invaluable when dealing with the switching effects.

To synthesize a controller, it is beneficial to first express the entire system model in a state space form. For this system, the state variables will be designated as the sphere's Y and Z positions, the velocity components $\dot{Y}$ and $\dot{Z}$, and currents in the four coils j_1, j_2, j_3, and j_4. The inputs for this system are the coil voltages v_1, v_2, v_3, and v_4. This results in the state vector

$$x = \left[\begin{array}{cccccccc} Y & \dot{Y} & Z & \dot{Z} & j_1 & j_2 & j_3 & j_4 \end{array} \right]^T, \quad (23)$$

and the input vector

$$u = \left[\begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \right]^T. \quad (24)$$

It will be assumed that the measured outputs are the sphere positions Y and Z, and the four coil currents. As can be seen from (12) and (13), this system may be represented as the two linear systems of sphere dynamics and electrical dynamics coupled by the nonlinear magnetic force terms. In addition any unknown drag force which deviates from the linear Stokes' static drag may be treated as a disturbance. This leads to the state space representation of

$$\dot{x} = [A]x + f(x) + [B]u + [D], \quad (25a)$$

$$y = [C]x, \quad (25b)$$

where

$$[A] = \left[\begin{array}{cc} A_s & [\emptyset]_{4\times4} \\ [\emptyset]_{4\times4} & -L^{-1}R \end{array} \right], \quad [B] = \left[\begin{array}{c} [\emptyset]_{4\times4} \\ L^{-1} \end{array} \right],$$

$$A_s = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & \frac{-6\pi r\mu}{m} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{-6\pi r\mu}{m} \end{array} \right], \quad [D] = \left[\begin{array}{c} 0 \\ d_Y \\ 0 \\ d_Z \\ [\emptyset]_{4\times1} \end{array} \right],$$

$$[C] = \left[\begin{array}{cc} C_s & [\emptyset]_{4\times4} \\ [\emptyset]_{4\times4} & [I]_{4\times4} \end{array} \right], \quad C_s = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right],$$

$$f(x) = \left[\begin{array}{c} 0 \\ \frac{4\pi\mu_0 r^3}{3m} \frac{M}{\sqrt{x^T\Gamma x}} x^T[\mathbf{K}_{FM,Y}]x \\ 0 \\ \frac{4\pi\mu_0 r^3}{3m} \frac{M}{\sqrt{x^T\Gamma x}} x^T[\mathbf{K}_{FM,Z}]x \\ [\emptyset]_{4\times1} \end{array} \right],$$

$$[\mathbf{K}_{FM,S}] = \left[\begin{array}{cc} [\emptyset]_{4\times4} & [\emptyset]_{4\times4} \\ [\emptyset]_{4\times4} & [\mathbf{k}_{FM,S}] \end{array} \right],$$

$$\Gamma = \left[\begin{array}{cc} [\emptyset]_{4\times4} & [\emptyset]_{4\times4} \\ [\emptyset]_{4\times4} & [G] \end{array} \right].$$

Note that in (25), $[\emptyset]_{m\times n}$ represents an $m \times n$ matrix of zeros, $[I]_{m\times m}$ represents an $m \times m$ identity matrix, and d_S, $S = X, Y, Z$, represents the unknown component flow disturbances. Equation (25) is an eighth order nonlinear state space model which may be used for controller synthesis and design. As can be seen, the system is linear in the input u but contains rather formidable nonlinear terms describing the field forces. With the system now represented in a state space form, the formulation of the field components as they pertain to the coil geometries and the calculation of the mutual inductances must be examined in greater detail. This will be explored in the future.

V. Conclusion

A model describing a system to manipulate the position of a magnetic sphere in a fluid using stationary electromagnets has been developed. From an electromagnetic standpoint, the "attractive only" nature leads to an underdetermined system with multiple solutions. By reducing the number of active coils to only those necessary to produce the desired force, solution of the coil currents is feasible. However, reduction of the set to the minimal number of coils can result in regions of the trajectory space where certain directions of force may not be capable due to the narrowing of the force angle range. Nevertheless, if this characteristic is firmly understood it may be possible to design trajectories avoiding this condition. A technique for determining the coil combination candidates for a required force has also been developed. By examining the sign definiteness of a coil combination's superpositioned geometric field function, one may establish whether a current solution for that coil combination even exists. In this work, this quantitiy is referred to as the $[\mathbf{\Omega}]$ matrix, and is proportional to a desired magnetic force at a given location in the plane. An interesting aspect of this technique is that it does not involve explicit calculation of the currents themselves.

The $[\boldsymbol{\Omega}]$ matrix is a function of the desired magnetic force vector, geometric characteristics of the coils, and position of the sphere. The current flowing in the coils is absent from the $[\boldsymbol{\Omega}]$ matrix formulation.

Upon examination it is clear that with respect to the fluid drag force being represented as Stokes flow in the cited literature, this approximation may not be entirely accurate for all of the envisioned applications. When the target size approaches larger dimensions, as would be the case in the employment of a micro-robot or device, the Reynolds number describing the flow about the object may deviate from the linear region of drag force. Additionally the larger object may cause the assumption of an unbounded free stream to break down, requiring the model to account for channel flow effects. In these cases, accurate calculation or prediction of these dynamics may require modification of the model describing fluid drag similar to equations (6) and (7). In some flow conditions, a numerical model may be required to achieve the desired accuracy. Even with improved model precision, open loop control of this type of system may prove to be inadequate or even impossible to achieve due to the inherent instability present when using an electromagnet as an actuation device. With that said, further exploration into closed loop control as a means to overcome flow disturbances and innate modeling errors is necessary. The state space model presented in this paper is a firm footing to begin the closed loop system dynamic analysis. The solution developed in this paper is valid for the 2-D planar motion problem, however it is conceivable to expand this work to the full 3-D motion system. An extension into this case would entail increasing the reduced minimal coil set from two coils to three. In addition the current solution would be modified from an intersection of curves similar to conic sections to the intersection of quadric surfaces. While this may prove to be analytically and computationally challenging, the broadening of this technique to the 3-D case is believed achievable.

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