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Control of Contact Problem in Constrained Euler-Lagrange Systems

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Abstract. In this paper, we investigate the contact problem for constrained Euler-Lagrange systems. We model the constrained equations as a set of non-smooth differential equations depending on whether the system lies on the constraint surface (active phase) or the system repeatedly makes and loses contact with the constraint surface (transition phase). We concentrate on the initial condition problem for the transition phase, i.e., the system hits the constraint with a non-zero normal velocity. We state a generic impact model to describe the impact behavior. The control laws are designed such that during the contact/non-contact phase of the system there is a potential force field that will always direct the system towards the constraint.

1 Introduction

In this paper, we consider systems described by Euler-Lagrange equations with a single unilateral constraint. Let the kinetic and potential energy of the Euler-Lagrange system be given by $\mathcal{K}(x,\dot{x}) = \frac{1}{2}\dot{x}^T M(x)\dot{x}$ and $\mathcal{P}(x)$ where $x \in \mathbf{R}^n$ is the generalized position, $\dot{x} \in \mathbf{R}^n$ is the generalized velocity, and $M(x) \in \mathbf{R}^{n\times n}$ is the symmetric positive definite inertia matrix. Let $u \in \mathbf{R}^n$ be the input forces, and $f_\mu(x,\dot{x}) \in \mathbf{R}^n$ denote the non-conservative forces resulting from the discontinuous elements like friction, dead zones etc. Let the single constraint be given by $\phi(x) - d \geq 0$, where d is a constant. In free motion of the system, the constraint takes the strict inequality form. When in contact with the surface, the constraint forces on the system are given by $f_\phi(x,\lambda) = g_\phi^T(x)\lambda$, where $g_\phi(x) := \partial\phi(x)/\partial x$ and λ is the Lagrange multiplier associated with the constraint. The multiplier is zero in free motion and nonzero when the system lies on the constraint surface. The Lagrangian for the system is $\mathcal{L}(x,\dot{x}) = \mathcal{K}(x,\dot{x}) - \mathcal{P}(x)$. The motion of the constrained system is described by Euler-Lagrange equations.

Assuming the constrained Euler-Lagrange system is as detailed above, the problem that is investigated in this paper can be stated as follows: given non-zero initial conditions satisfying the constraint surface, we want to first characterize the motion of the constrained system, design stable control laws for the transition phase(where the system makes and loses contact repeatedly) and the active phase (constrained motion of the system on the surface) with the aim of minimizing the bouncing behavior in the transition phase. Further we want to assure that the overall closed-loop system taking into account switching between both phases is stable. Another issue is when the system loses contact during constrained motion due to some unknown disturbances, then the controllers should be designed to bring back the system onto the constrained surface. A complete version of the paper including the literature review and proofs can be obtained by e-mail to the author or at the web site http://kirthi.mae.okstate.edu/~pagilla/.

2 Constrained Dynamics

The task space vector (x) can be partitioned as $x^T = (x_1, x_2^T)$, where $x_1 \in \mathbf{R}^1$ and $x_2 \in \mathbf{R}^{n-1}$. A canonical transformation, $\psi(\cdot) : (x_1, x_2) \to (q = x_1 - \sigma(x_2), r = x_2)$, where $q \in \mathbf{R}^1$ and $r \in \mathbf{R}^{n-1}$, can be performed such that the constraint in the transformed coordinate is represented by $q = 0$. The existence of such a transformation is well documented in the literature [2, 3]. Let the kinetic energy in the transformed coordinates be: $\mathcal{K}_*(q,\dot{q},r,\dot{r}) = (1/2)\{a(q,r)\dot{q}^2 + 2\dot{q}b^T(q,r)\dot{r} + \dot{r}^T A(q,r)\dot{r}\}$. In this expression, the mass matrix in the transformed coordinates is partitioned appropriate to the size of the transformed coordinates (q,r). Let $\mathcal{P}_*(q,r)$ represent the potential energy in the transformed coordinates, then the transformed Lagrangian is $\mathcal{L}_*(q,r,\dot{q},\dot{r}) = (1/2)(a\dot{q}^2 + 2\dot{q}b^T\dot{r} + \dot{r}^T A\dot{r}) - \mathcal{P}_*(q,r)$.

Let S denote the task space in the transformed coordinates, then S can be subdivided into sets S_c, S_u, and S_f i.e., $S = S_c \cup S_u \cup S_f$, where $S_u := \{(q,r) \ : \ q > 0\}$, $S_f := \{(q,r) \ : \ q < 0\}$ and

$$S_c := (S_{ct} := \{(q,r) : q = 0, \dot{q} \neq 0\}) \cup (S_{ca} := \{(q,r) : q = 0, \dot{q} = 0\})$$

Notice that $S_c = S_{ct} \cup S_{ca}$. The motivation for this sub-division stems from the fact that the mechanical system can impact the surface with a non-zero velocity, causing a jump condition in the velocity $\dot{q}$. With this partition of the transformed task space, it can be seen that the mechanical system lies on one side of the constraint surface $q = 0$. Without loss of generality it can be assumed that the mechanical system lies in S_u. The constrained system is now given by the transformed Lagrangian $\mathcal{L}_*(q,r,\dot{q},\dot{r})$; the input forces are given by $(u_q, u_r) := T^T(r)u$; the non-conservative forces are given by $(f_{\mu q}, f_{\mu r}) := T^T(r)f_\mu$; and the constraints on the system are given by $q \geq 0$. The constrained forces in the transformed coordinates are $(f_{\phi q}, 0) := T^T(r)f_\phi$. Notice that the last $n-1$ elements of the vector $T^T(r)f_\phi$ are zeros, due to the fact that the direction of the velocity $\dot{q}$ is normal to the constraint surface. Denote $c := \partial\mathcal{L}_*/\partial q + \dot{a}\dot{q} - f_{\mu q}$; $C := \partial\mathcal{L}_*/\partial r + \dot{A}\dot{r} - f_{\mu r}$. Notice that $c(q,\dot{q},r,\dot{r}) \in \mathbf{R}^1$ and $C(q,\dot{q},r,\dot{r}) \in \mathbf{R}^{n-1}$. Using Lagrange's equation of motion and the sub-division of the transformed task space, the following equations describe the system constrained unilaterally:

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If $(q,r) \in S_u$, the equations are:

$$\begin{aligned} a(q,r)\ddot{q} + c(q,\dot{q},r,\dot{r}) &= u_q \\ A(q,r)\ddot{r} + C(q,\dot{q},r,\dot{r}) &= u_r \end{aligned} \qquad (2.1)$$

If $(q,r) \in S_{ct}$, then the jump condition for (2.1) is

$$\dot{q}_+ = \mathcal{D}(q,\dot{q}_-) \qquad (2.2)$$

If $(q,r) \in S_{ca}$, then constraint forces are greater than zero and the equations are:

$$\begin{aligned} c(0,0,r,\dot{r}) &= u_q + f_q \\ A(q,r)\ddot{r} + C(0,0,r,\dot{r}) &= u_r \end{aligned} \qquad (2.3)$$

In (2.2), $\dot{q}_+$ and $\dot{q}_-$ represent the impact velocity and the rebound velocity respectively, and $\mathcal{D}(\cdot)$ represents an operator which maps the impact velocity to the rebound velocity. This operator can take several forms depending on the choice of the impact model for the constraint surface.

3 Controller Design and Stability

The control objective is as follows: Given non-zero initial condition, i.e., $q(0) = 0$ and $\dot{q}(0) \neq 0$, we want to design stable control algorithms for constrained non-smooth dynamic equations (2.1) to (2.3), for regulation of $q(t)$ and $\dot{q}(t)$ to zero during transition, tracking of desired forces during active phase and tracking of $r(t)$ to follow the profile of the constraint surface during both phases. We assume that the desired trajectories for $r(t)$ and $f_q(t)$ be $r^d(t)$ and $f_q^d(t)$ respectively, for tangential motion and normal contact force.

For ease of expressing the controllers and the closed loop equations the following errors are defined: $e_{vq} = \dot{q} + \lambda_q q; e_r = r(t) - r^d(t)$; $e_{fq} = f_q(t) - f_q^d(t); e_{vfq} = \lambda_{fq}\int_0^t e_{fq}(\omega)d\omega$, where e_r is the tangential position tracking error, e_{vq}, e_{vr} are the velocity reference errors in the variables q and r respectively, and e_{fq}, e_{vfq} are the force error and the reference force error respectively. Before designing the control algorithms the following non-smooth positive definite Lyapunov functions are defined: $V_q(e_{vq}) := |e_{vq}|$, $V_r(e_{vr}) := \sum_{i=1}^{n-1}|e_{vr}|$, and $V_{fq}(e_{vfq}) := |e_{vfq}|$. The following control laws are proposed for the constrained dynamics given by (2.1) to (2.3),

If $(q,r) \in S_u \cup S_{ct}$, then

$$\begin{aligned} u_q = c(q,r,\dot{q},\dot{r}) &+ a(q,r)[\ddot{q}^d - \lambda_q \dot{e}_q] \\ &- k_{qc}\nabla V_q(e_{vq}) - \alpha_c q e_{vq} - \alpha_c \frac{\partial \mathcal{P}_q(q)}{\partial q} \end{aligned} \qquad (3.1)$$

$$u_r = C(q,r,\dot{q},\dot{r}) + A(q,r)[\ddot{r}^d - \lambda_r \dot{e}_r] - k_{rc}\nabla V_r(e_{vr}) \qquad (3.2)$$

If $(q,r) \in S_{ca}$, then

$$u_q = c(0,r,0,\dot{r}) + f_q^d - k_{fq}\nabla V_{fq}(e_{vfq}) \qquad (3.3)$$

$$\begin{aligned} u_r = C(0,r,0,\dot{r}) &+ A(0,r)[\ddot{r}^d - \lambda_r \dot{e}_r] \\ &+ f_r^d - k_{fr}\nabla V_{fr}(e_{vfr}) - k_{ra}\nabla V_r(e_{vr}) \end{aligned} \qquad (3.4)$$

where $k_{qc}, k_{rc}, k_{ra}, k_{fq}$, k_{fr}, α_c are positive constants, and $\mathcal{P}_q = \frac{1}{2}q^2$. ∇V represents the generalized gradient[1] of the non-smooth Lyapunov function V. Notice that the control law (3.1) contains a nonlinear term and a potential force field term.

Theorem 3.1 (Main Theorem) *For the differential equations describing the motion of the constrained Euler-Lagrange system, using the control laws given by (3.1) to (3.4), the following are true:*

1. *In the contact/non-contact phase (transition), the bounces settle down in finite time, i.e., q and $\dot{q}$ converge to zero and further the tracking error, e_r converges to zero.*
2. *In the constrained motion phase (active), the force error(e_f) and tangential position tracking error(e_r) converge to zero.*

4 Discussions and Conclusions

There are several important issues that can constitute problems for future research. The upper bound on impact velocity when the system sticks to the surface, is still unknown. This bound directly determines when the active phase starts. Starting the active phase before the impact velocities become significantly small can jeopardize closed-loop stability. This is because the force transients due to impact forces are very high, and when used in feedback can de-stabilize the system. If we have a highly compliant surface than this may not be a big issue. But in many practical applications such as surface finishing using robots, the surface to be machined is very rigid, and hence high impact forces for small impact velocities. In general frictional behavior, both tangential and normal, at the contact coupled with the impact model have to be studied in detail. It should be interesting to analyze existence of limit cycles and their stability for such systems. Choice of the controller gains will depend on how fast we want the transients to settle down upon contact. Also, we have not discussed robustness to unknown system parameters, that could be an area of future research for this formulation.

In this paper, we have described the dynamics of the contact problem for Euler-Lagrange systems by a set of non-smooth differential equations. We have designed stable control algorithms for such systems. Our approach is unique in the sense that the control laws do not depend on the impact model and also the controller includes a potential force field term which assures return of the system onto the surface in the event of loss of contact during active phase due to unknown disturbances. Further, the non-smooth model for the constrained dynamics and subsequent non-smooth analysis depicts the natural behavior of the contact problem for Euler-Lagrange systems.

It should be noted that the reference list below is quite incomplete. For a complete version of this paper, please e-mail the author or check the web site: http://kirthi.mae.okstate.edu/~pagilla/.

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