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CONTROL OF MECHANICAL SYSTEMS SUBJECT TO UNILATERAL CONSTRAINTS*

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Abstract

In this work we consider the problem of control of mechanical systems subject to unilateral constraints. Impulsive forces arise whenever the constraints become active and these forces give rise to nonsmooth dynamics. The dynamics of the system is defined by a set of differential equations with discontinuous righthand side using Hamilton's equations of motion. A nonlinear transformation is applied and the dynamics of the system is written in two sets of differential equations in the transformed coordinates. Three different phases (inactive, transition and active) for the system are formulated depending on the activation/deactivation of the constraints. A discontinuous controller is designed for the three phases for tracking the desired trajectories of the system. Stability analysis is conducted for all the phases using tools like Filippov's differential inclusions, nonsmooth Lyapunov analysis and generalized gradients. We give an illustrative example for the theory developed.

1 Introduction

In many mechanical systems interacting with an environment, there are moments of time when they experience a change of state abruptly. One common application in industry is a robot following an external surface. In many of these type of applications, first the robot moves in free space, then makes contact with a surface and follows constrained motion for a specified time and leaves the surface. A mathematical model for such systems is a set of differential equations subject to unilateral constraints. When the robot is in free space then the surface is represented as strict inequality constraints and the constraints are inactive, and when the robot is in contact with the surface the constraints are active and expressed as equality constraints. Whenever the inactive constraints become active, impulsive forces are generated which give rise to nonsmooth dynamics. The impulsive forces on the system have to vanish before contact force on the surface can be controlled. This phase where the inactive constraints become active and there are impulses in the system is called a transition phase. Effective control algorithms have to be designed for each phase of the system.

Considerable research has been done in control of robots in constrained motion; see [7, 10] and references therein. Much of this research has been based on the assumption that the robot is already in contact with the external environment. In many of the industrial applications the mechanical system is in free motion before constrained motion starts. The transition from free motion to constrained motion leads to impulsive forces on the system. Stability and control of task transition for robots has been considered in [12] for compliant environment, wherein the transition are assumed to take place smoothly. Impact minimization by using redundant degrees of freedom in the robots has been considered in [13]. Nonsmooth Lyapunov analysis using Filippov's differential inclusions and generalized gradients is shown in [5, 6].

The formulation given in this paper covers a broad class of control problems for mechanical systems interacting with environment including collision, impact etc. We formulate the nonsmooth equations of motion using the Hamiltonian framework. The equations of motion are expressed as first order differential equations in generalized coordinates and generalized momentum variables. We transform the generalized momentum variables using a nonlinear transformation. Three phases of motion are formulated depending on the activation/deactivation of the constraints, these are the inactive, transition and the active phases. A different set of differential equations describes the dynamics of the system in each phase. Discontinuous control laws are proposed for each phase. Stability analysis is conducted for the proposed control laws for different phases using tools like Filippov's differential inclusions, generalized gradients and nonsmooth Lyapunov analysis.

The rest of the paper is organized as follows: In section 2, we give the basic equations of the system, explain an impact model and derive differential equation for different phases of the system. In section 3 some mathematical preliminaries are given and control laws are proposed. Stability of the proposed controllers is also shown. We discuss the practical limitations associated with the problem and the future research that has to be conducted in section 4. Section 5 contains the conclusions.

2 Basic Equations

By mechanical systems, we mean systems with kinetic and potential energy functions of the form, $\mathcal{K}(q,\dot{q}) = \frac{1}{2}\dot{q}^\top M(q)\dot{q}$ and $\mathcal{P}(q)$, where $q \in \Re^n$ is the generalized position, $\dot{q} \in \Re^n$ is the generalized velocity, and $M(q) \in \Re^{n\times n}$ is the symmetric positive definite mass matrix. The system is subject to unilateral constraints of the form

$$\Phi(q) \leq 0 \tag{2.1}$$

where $\Phi^\top(q) = [\phi_1(q), \phi_2(q), \cdots, \phi_m(q)], m \leq n$. The map $\Phi(\cdot): \Re^n \to \Re^m$ is assumed to be sufficiently smooth and the Jacobian associated with it, $G_c(q) = \frac{\partial\Phi(q)}{\partial q}$, has full row rank. It is also assumed without loss of generality that the rows of $G_c(q)$ are orthogonal. The Lagrangian of the constrained mechanical system is a function

$$\mathcal{L}^*(q,\dot{q}) = \mathcal{K}(q,\dot{q}) - \mathcal{P}(q) + \Phi^\top\lambda \tag{2.2}$$

We define two cases depending on the value of $\Phi(q)$, (a) when $\Phi(q) < 0$, then $\lambda = 0$ and the constraint is said to be inactive,

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(b) when $\Phi(q) = 0$, then $\lambda \geq 0$ and the constraint is active. Impact forces arise whenever an inactive constraint becomes active and these forces are impulsive in nature and give rise to possible discontinuities in the velocity variable [1]. The generalized momentum variables are defined as $p = M(q)\dot{q}$. The Hamiltonian for the system is a function

$$\mathcal{H}(q,p) = p^\top \dot{q} - \mathcal{L}^*(q,\dot{q}) \tag{2.3}$$

and using the Hamilton's equations of motion we obtain,

$$\begin{aligned} \dot{q} &= M^{-1}(q)p \\ \dot{p} &= -\frac{1}{2}(I \otimes p^\top)\frac{\partial M^{-1}}{\partial q}p + \tau + \tau_g + G_c^\top \lambda \end{aligned} \tag{2.4}$$

where $\tau_g = \frac{\partial \mathcal{P}}{\partial q}$, $\otimes$ represents the Kronecker product, and I_n represents $n \times n$ identity matrix. Notice that $\frac{\partial M^{-1}}{\partial q}$ is a $n^2 \times n$ matrix, and $(I_n \times p^\top)$ is a $n \times n^2$ matrix. Also, in the presence of the unilateral constraints the system state, q, lies in the region, $E := \{q \in \Re^n : \Phi(q) \leq 0\}$. Another region where all the constraints are equal to zero is given as $S := \{q \in E : \Phi(q) = 0\}$. When the constraints are inactive the system state q lies in the region $E - S$ and q is in S when the constraint is active.

2.1 Change of Coordinates

The equations of motion (2.4) are expressed in terms of the generalized coordinate q and the generalized momenta p. The differential equations do not have any transparent relationship with the constraints, except that the motion in these coordinates of the mechanical system is such that it does not violate these constraints. We would like to obtain a new set of coordinates that can be explicitly expressed in some sense with respect to the directions that are tangential and normal to the surface S. To obtain these new set of coordinates we consider the following transformation of the momentum variable p, $\theta = G(q)p$, where $\theta \in \Re^n$ are the transformed generalized momenta and $G^\top(q) = [G_c^\top, G_u^\top]$. The matrix $G_u(q)$ is $(n-m) \times n$ and its columns are orthogonal and obtained by completing the basis. Notice that with this construction, $G(q)$ is an orthogonal transformation matrix. Now, using the transformed momenta the equations of motion (2.4) can be written as

$$\begin{aligned} \dot{q} &= M^{-1}(q)G^{-1}(q)\theta \\ \dot{\theta} &= -\frac{1}{2}G(q)(I_n \otimes \theta^\top G^{-\top})\frac{\partial M^{-1}}{\partial q}G^{-1}\theta \\ &\quad + \dot{G}G^{-1}\theta + G\tau + G\tau_g + GG_c^\top\lambda \end{aligned} \tag{2.5}$$

The equations (2.5) can also be written in short as

$$\dot{\theta} + N(q,\theta) = G(q)\tau + G(q)G_c^\top(q)\lambda \tag{2.6}$$

Noting that $[G(q)G_c^\top(q)]^\top = [G_cG_c^\top, O_{m\times(n-m)}]$, where $O_{m\times(n-m)}$ is a zero matrix of size $m \times (n-m)$, the equations of motion (2.6) can be split up into two sets of differential equations as follows

$$\dot{\theta}_c + N_c(q,\theta) = \tau_c + J_c^\top(q)\lambda_c \tag{2.7}$$

$$\dot{\theta}_u + N_u(q,\theta) = \tau_u \tag{2.8}$$

where $\theta^\top = [\theta_c^\top, \theta_u^\top]$, $N_c(q,\theta) \in \Re^m$ and $N_u(q,\theta) \in \Re^{n-m}$ are the first m and last $(n-m)$ components of the n vector $N(q,\theta)$ and $J_c^\top(q) = G_c(q)G_c^\top(q)$. τ_c and τ_u are the first m and last $(n-m)$ components of the vector $G(q)\tau$ respectively.

The first set of differential equations (2.7) represent motion of the mechanical system normal to the surface S, and the second set of differential equations represent motion in the tangential directions of the surface S. Notice that, the motion tangential to the surface does not involve any constraint force terms, and the motion in the constrained directions involves the constraint force terms when the constraints are activated.

2.2 Impact Model

There is a discontinuous velocity change whenever an inactive constraint becomes active [1]. The discontinuous velocity change is caused by an impulsive force on the system. The impulsive forces depends on the impact model. We choose a simple rigid body collision to model impact. Consider a collision between two rigid bodies, the relationship between velocities before impact $\{v_1, v_2\}$ and after impact $\{v_1', v_2'\}$ is given by $\varepsilon = -\frac{(v_1'-v_2')}{(v_1-v_2)}$, where ε is a non-negative constant called the coefficient of restitution. This equation holds if the volume of contact is small. The value of ε depends on the type of collision, $\varepsilon = 1$ for perfectly elastic impact and $\varepsilon = 0$ for perfectly plastic impact. A smaller value of ε means a more loss of mechanical energy due to the collision. If we consider the second particle is stationary before and after impact, we can write

$$\Delta v_1 := v_1' - v_1 = -(1+\varepsilon)v_1 \tag{2.9}$$

We will use this collision model for the mechanical systems we consider, to compute the velocity changes during impacts, i.e. when the constraints are activated. Generally, impacts are treated as very large forces acting over a short duration of time. If we assume that the impact occurs over an infinitesimally small period of time, then (i) all velocities remain finite and (ii) there is no change in the position of the system. If Δt is the duration of collision and $F(\omega)$ is impact force during collision, then the force impulse F_I due to the impact at time t_* is

$$F_I = \lim_{\Delta t \to 0} \int_{t_*}^{t_*+\Delta t} F(\omega)d\omega. \tag{2.10}$$

Since the integration interval is of zero measure, $F(\omega)$ must take infinitely large values for F_I to be non-zero. So, $F(\omega)$ must be considered as a Dirac measure at time t_* with magnitude F_I. Expressing $F = F_I\delta_{t_*}$ makes the notion of large forces acting for a short time dissappear, as it allows one to separate the magnitude of impact force F_I and its distribution on time axis δ_{t_*} [11]. Now, integrating (2.7) over the interval t to $t + \Delta t$, we obtain,

$$(\theta_c^+ - \theta_c^-) = J_c^\top(q)\lambda_I \tag{2.11}$$

where λ_I is the magnitude of the impulsive force and the direction of which is opposite to the direction of $\dot{\theta}_c^-$. Similar to equation (2.9) we can write

$$\Delta\theta_c = -(1+\varepsilon)\theta_c^- \tag{2.12}$$

Combining (2.11) and (2.12),

$$\lambda_I = -(1+\varepsilon)J_c^{-\top}(q)\theta_c^- \tag{2.13}$$

Equations (2.12) gives the relationship between the velocity after impact and the velocity before impact, and equation (2.13) is the expression for the magnitude of the impact force acting on the system.

2.3 Mathematical Model for Control

In this section we develop a mathematical model for control considering all the possible scenarios the system (2.4) goes through when subjected to unilateral constraints of the form (2.1). We design the system to go through three phases of motion: (i) inactive phase (ii) transition phase (iii) active phase. The system can be in any one or more of these phases for any given control task. In the inactive phase the system is in free space and the constraints are strict inequality constraints. When the constraints are suddenly introduced then the system is in a transition motion, where in the velocities normal to the surface S

are non-zero and impact forces act on the system. So, in this transition phase the goal is to drive the velocities normal to the surface S to zero. In the active phase, motion control is applied for the coordinates, θ_u, which are tangential to the surface S, and contact force control is applied for the normal directions.

We also design the impact forces on the system during the transition phase as impulsive disturbances and express the first set of differential equations during that phase to be acted upon by these impulses. The equations of motion (2.7) and (2.8) can be written for the different phases as

Inactive phase :

$$\begin{aligned} \dot{q} &= M^{-1}(q)G^{-1}(q)\theta && (2.14)\\ \dot{\theta}_c &= -N_c(q,\theta)+\tau_c && (2.15)\\ \dot{\theta}_u &= -N_u(q,\theta)+\tau_u && (2.16) \end{aligned}$$

Transition phase :

$$\begin{aligned} \dot{q} &= M^{-1}(q)G^{-1}(q)\theta && (2.17)\\ \dot{\theta}_c &= -N_c(q,\theta)-D_c(\theta)\delta(\Phi(q))+\tau_c && (2.18)\\ \dot{\theta}_u &= -N_u(q,\theta)+\tau_u && (2.19) \end{aligned}$$

Active phase :

$$\begin{aligned} \dot{q} &= M^{-1}(q)G^{-1}(q)\theta && (2.20)\\ \tau_c &= N_c(q,\theta_u)-J_c^\top\lambda_c && (2.21)\\ \dot{\theta}_u &= -N_u(q,\theta_u)+\tau_u && (2.22) \end{aligned}$$

where $\delta^\top(\Phi(\theta)) = [\delta(\phi_1(\theta)), \delta(\phi_2(\theta), \cdots, \delta(\phi_m(\theta))]$, each term $\delta(\phi_i(\theta))$ means an impulse defined by the condition $\phi_i(\theta)=0$, $D_c(\theta)$ is the matrix of magnitudes of the impulsive forces, and λ_c is the contact force on the surface.

Remarks: Notice that the system has to follow a particular sequence of phases for a given task. For example, the system cannot go from inactive phase to active phase, when the constraints are activated suddenly, i.e. when the velocities normal to the surface S are non-zero.

3 Controller Design

The dynamics of a mechanical system subject to unilateral constraints is nonsmooth and is given by discontinuous differential equations (2.15) - (2.22). Filippov [2] developed a solution concept for differential equations with discontinuous right hand side. In [6] a calculus for computing Filippov's differential inclusion is presented and in [5] Filippov's differential inclusions and generalized gradients are used to show stability of nonsmooth systems using nonsmooth Lyapunov functions. We use a similar approaches as given in [6] and [5] to prove the stability of the discontinuous differential equations (2.15) - (2.22) under discontinuous control laws. The definition of the Fillipov's differential inclusion can be found in [2] and the definitions of the generalized directional derivative and the generalized gradient can be found in [3].

3.1 Control Laws:

Before proposing the discontinous control laws, we define the errors and the Lyapunov functions which will be used in the control laws. The different errors are given as:

$$\begin{aligned} e_c &:= q_1-q_{1d}\\ e_u &:= q_2-q_{2d}\\ e_f &:= \lambda_c-\lambda_{cd}\\ \theta_{ud} &:= Q_u\dot{q}_{2d}+Q_{c2}\dot{q}_{1d}\\ \theta_{cd} &:= Q_c\dot{q}_{1d}+Q_{c1}\dot{q}_{2d}\\ e_{vc} &:= (\theta_c-\theta_{cd})+Q_c\Lambda_c e_c+Q_{c1}\Lambda_u e_u\\ e_{vu} &:= (\theta_u-\theta_{ud})+Q_u\Lambda_u e_u+Q_{c2}\Lambda_c e_c \end{aligned}$$

where $q_{1d}\in\Re^m, q_{2d}\in\Re^{n-m}$, and $\lambda_{cd}\in\Re^m$ are the desired trajectories of q_1, q_2, and λ_c respectively. Notice that $q^\top = [q_1^\top, q_2^\top]$, $e^\top = [e_c^\top, e_u^\top]$, $\theta^\top = [\theta_c^\top, \theta_u^\top]$ and $\Lambda = diag(\Lambda_c, \Lambda_u)$. We assume that q_{1d} and q_{2d} are twice differentiable. The matrices $\Lambda_c\in\Re^{m\times m}$ and $\Lambda_u\in\Re^{n-m\times n-m}$ are positive definite gain matrices. The matrices $Q_c\in\Re^{m\times m}$, $Q_{c1}\in\Re^{m\times n-m}$, $Q_{c2}\in\Re^{n-m\times m}$, and $Q_u\in\Re^{n-m\times n-m}$ are obtained from the following decomposition of $G(q)M(q)$ as

$$G(q)M(q) =: Q(q) = \begin{bmatrix} Q_c(q) & Q_{c1}(q)\\ Q_{c2}(q) & Q_u(q)\end{bmatrix}$$

Now, the Lyapunov functions are given by,

$$V_c(e_{vc}) := \|e_{vc}\|_1 = \sum_{i=1}^{m}|e_{vc}^i| \qquad (3.1)$$

$$V_u(e_{vu}) := \|e_{vu}\|_1 = \sum_{i=1}^{n-m}|e_{vu}^i| \qquad (3.2)$$

$$V_f(e_{vf}) := \|e_{vf}\|_1 = \sum_{i=1}^{m}|e_{vf}^i| \qquad (3.3)$$

Consider the following control laws for each phase of motion

Inactive phase :

$$\begin{aligned} \tau_c &= \widehat{N}_c(q,\theta)-k_{ic}(q,\theta,q_d,\dot{q}_d,\ddot{q}_d)\nabla V_c(e_{vc}) && (3.4)\\ \tau_u &= \widehat{N}_u(q,\theta)-k_{iu}(q,\theta,q_d,\dot{q}_d,\ddot{q}_d)\nabla V_u(e_{vu}) && (3.5) \end{aligned}$$

Transition phase :

$$\begin{aligned} \tau_c &= \widehat{N}_c(q,\theta)-k_{tc}(q,\theta,q_d,\dot{q}_d,\ddot{q}_d)\nabla V_c(e_{vc}) && (3.6)\\ \tau_u &= \widehat{N}_u(q,\theta)-k_{tu}(q,\theta,q_d,\dot{q}_d,\ddot{q}_d)\nabla V_u(e_{vu}) && (3.7) \end{aligned}$$

Active phase :

$$\begin{aligned} \tau_c &= \widehat{N}_c(q,\theta)+k_f(q,\theta,q_d,\dot{q}_d,\ddot{q}_d)\nabla V_f(e_{vf})-J_c^\top\lambda_c && (3.8)\\ \tau_u &= \widehat{N}_u(q,\theta)-k_{au}(q,\theta,q_d,\dot{q}_d,\ddot{q}_d)\nabla V_u(e_{vu}) && (3.9)\\ & && (3.10) \end{aligned}$$

where $\widehat{N}_c(q,\theta)$ and $\widehat{N}_u(q,\theta)$ are the known part of the vectors, $N_c(q,\theta)$ and $N_u(q,\theta)$. The gains $k_{ic}(\cdot)$, $k_{iu}(\cdot)$, $k_{tu}(\cdot)$, $k_{tu}(\cdot)$, $k_f(\cdot)$ and $k_{au}(\cdot)$ will be determined during the stability analysis. $\nabla V_c(e_{vc})$, $\nabla V_u(e_{vu})$ and $\nabla V_f(e_{vf})$ are the gradients of the Lyapunov functions $V_c(e_{vc})$, $V_u(e_{vu})$ and $V_f(e_{vf})$ respectively.

3.2 Stability Analysis:

Inactive phase : Substituting the control (3.1) and (3.2) into the equations (2.15) and (2.16), the closed loop equations for this phase can be written as:

$$\begin{aligned} \dot{\theta}_c &\in -k_{ic}K[\nabla V_c(e_{vc})]+K[\widehat{N}_c(q,\theta)-N_c(q,\theta)]\\ \dot{\theta}_u &\in -k_{iu}K[\nabla V_u(e_{vu})]+K[\widehat{N}_u(q,\theta)-N_u(q,\theta)] \end{aligned}$$

where $K[f(\cdot)]$ is Fillipov differential inclusion of the function $f(\cdot)$; see [2, 6] for the definition. Using the definition of errors, the derivative of e_{vc} and e_{vu} can be written as

$$\begin{aligned} \dot{e}_{vc} &\in -k_{ic}\partial V_c(e_{vc})+K[\widetilde{N}_c]+Q_c\Lambda_c\dot{e}_c+Q_{c1}\Lambda_u\dot{e}_u\\ &+ \dot{Q}_c\Lambda_c e_c+\dot{Q}_{c1}\Lambda_u e_u-\dot{\theta}_{cd} && (3.11)\\ \dot{e}_{vu} &\in -k_{iu}\partial V_u(e_{vu})+K[\widetilde{N}_u]+Q_u\Lambda_u\dot{e}_u+Q_{c2}\Lambda_c\dot{e}_c\\ &+ \dot{Q}_u\Lambda_u e_u+\dot{Q}_{c2}\Lambda_c e_c-\dot{\theta}_{ud} && (3.12) \end{aligned}$$

From the above equations recalling Filippov's solution, e_{vc} and e_{vu} are absolutely continuous and from theorem (2) of [5], the Lyapunov functions V_c and V_u are absolutely continuous and their derivatives exist almost everywhere except for a set of measure zero. Now, the following theorem gives stability of the solutions of the equations (3.7) and (3.8). The proof of which is given in the appendix I.

Theorem 3.1 *With the choice of the Lyapunov equations (3.1) and (3.2) and the closed loop equations (3.7) and (3.8), the reference velocities $\dot{e}_{vc}$ and $\dot{e}_{vu}$ converge to zero, if the gains $k_{ic}(\cdot)$ and $k_{iu}(\cdot)$ are chosen as following:*

$$\begin{aligned} k_{ic} &= \alpha_{ic} + \eta_{ic} \\ &+ \|Q_c\Lambda_c\dot{e}_c + Q_{c1}\Lambda_u\dot{e}_u + \dot{Q}_c\Lambda_c e_c + \dot{Q}_{c1}\Lambda_u e_u - \dot{\theta}_{cd}\| \\ k_{iu} &= \alpha_{iu} + \eta_{iu} \\ &+ \|Q_u\Lambda_u\dot{e}_u + Q_{c2}\Lambda_c\dot{e}_c + \dot{Q}_u\Lambda_u e_u + \dot{Q}_{c2}\Lambda_c e_c - \dot{\theta}_{ud}\| \end{aligned}$$

where $\alpha_{ic} > 0, \alpha_{iu} > 0, \eta_{ic} := \sup\{\|\eta_c\| : \eta_c \in K[\widetilde{N}_c]\}$ and $\eta_{iu} := \sup\{\|\eta_u\| : \eta_u \in K[\widetilde{N}_u]\}$.

Transition phase : Consider the equations (2.18) and (2.19) for motion in the transition phase. Using the control law (3.7), the stability of closed loop for equation (2.19) can be shown using the similar analysis as in the inactive phase. Now, for the constrained equations (2.18), we first give the following lemma, which shows that during impulses the Lyapunov function $V_c(e_{vc})$ is decreasing.

Lemma 3.1 *Suppose the velocities before and after the pth impulse are e_{vc}^{p+} and e_{vc}^{p-}, and the Lyapunov function $V_c(e_{vc}) = \|e_{vc}\|_1$, then*

$$V_c^{p+} - V_c^{p-} \leq (\varepsilon - 1)\|\theta_c^{p-}\|_1 \tag{3.13}$$

Proof: From the definition of the Lyapunov function we have

$$\begin{aligned} V_c^{p+} &:= \|(\theta_c^{p+} - \theta_{cd}^{p+}) + Q_c^{p+}\Lambda_c e_c^{p+} + Q_{c1}^{p+}\Lambda_u e_u^{p+}\|_1 \\ &\leq \|\theta_c^{p+}\|_1 + \|Q_c^{p+}\Lambda_c e_c^{p+} + Q_{c1}^{p+}\Lambda_u e_u^{p+} - \theta_{cd}^{p+}\|_1 \end{aligned}$$

$$\begin{aligned} V_c^{p-} &:= \|(\theta_c^{p-} - \theta_{cd}^{p-}) + Q_c^{p-}\Lambda_c e_c^{p-} + Q_{c1}^{p-}\Lambda_u e_u^{p-}\|_1 \\ &\leq \|\theta_c^{p-}\|_1 + \|Q_c^{p-}\Lambda_c e_c^{p-} + Q_{c1}^{p-}\Lambda_u e_u^{p-} - \theta_{cd}^{p-}\|_1 \end{aligned}$$

Noting that during impact the desired position and velocities remain constant and actual position does not change, we have

$$V_c^{p+} - V_c^{p-} \leq \|\theta_c^{p+}\|_1 - \|\theta_c^{p-}\|_1$$

But we know from (2.12) that $\theta_c^{p+} = -\varepsilon\theta_c^{p-}$, substituting this in the above equation, we obtain

$$V_c^{p+} - V_c^{p-} \leq (\varepsilon - 1)\|\theta_c^{p-}\|_1$$

◇

Notice that for $\varepsilon = 0$, the impact is plastic and there is no transition phase. For $\varepsilon = 1$, the Lyapunov function remains constant during the impact. If $\varepsilon < 1$ then the Lyapunov function is decreasing during the impact by the above lemma. Also, the Lyapunov function is decreasing in between the impacts. Using the same notation as in lemma (3.1), between adjacent impacts we obtain, $\|\theta_c^{p+}\|_1 \leq \|\theta_c^{(p+1)-}\|_1$. Thus, for an integer $r \geq p$, we obtain the following relationship,

$$\|\theta_c^{r+}\|_1 \leq (\varepsilon)^{r-p}\|\theta_c^{p+}\|_1 \tag{3.14}$$

Now the following theorem gives the stability of the transition phase,

Theorem 3.2 *Suppose for the differential equations given in (2.18), if the control law is given by (3.6), and the Lyapunov function is given as in (3.1), the reference velocity e_{vc} converges to zero, and as a result both e_c and $\dot{e}_c$ converge to zero.*

Proof: The closed loop equations without the impulsive forces are the same as (3.11). Since from theorem (3.1), the derivative of the Lyapunov function between adjacent impulses is negative, so the Lyapunov function is decreasing in this region. Also, during the impulses the Lyapunov function is decreasing as shown from lemma (3.1) and equation (3.14) for $\varepsilon < 1$. Therefore the Lyapunov function is decreasing at the instance of the impulses and also during the non-occurence of the impulses. Hence, we conclude that $e_{vc} \to 0$ and from the definition of e_{vc} and from the analysis of the proof of theorem (3.1), both e_c and $\dot{e}_c$ converge to zero. ◇

Although, we have shown the stability of the proposed control law for the transition phase, the performance is severely limited due to the presence of impulsive forces. We would like to reject the effect of these impulsive disturbances in some way so that we get better performance in the transition phase. Since, we know the estimate of the impulsive forces, we reject these over a finite period of time T_i after the impact occurs by increasing the gain k_{tc} smoothly. Suppose λ_{Ii} is an estimate of the impact force in the ith constrained direction, then we choose a function $f_i(t) \in L_1 \cap L_\infty$, such that it is zero at the instance of the impact and the area formed by the function $f_i(t)$ over T_i is equal to λ_{Ii}. By rejecting disturbances this way we do not have spikes in the control torques at the instances of impact.

$$\gamma_{ci} = \int_0^{T_i} f_i(t)dt$$

Let $\gamma_c^\top = [\gamma_{c1}, \gamma_{c2}, \cdots, \gamma_{cm}]$, then the gain k_{tc} is given by

$$\begin{aligned} k_{tc} = \alpha_{tc} + \eta_{tc} + \|Q_c\Lambda_c\dot{e}_c + Q_{c1}\Lambda_u\dot{e}_u \\ + \dot{Q}_c\Lambda_c e_c + \dot{Q}_{c1}\Lambda_u e_u - \dot{\theta}_{cd}\| + \|\gamma_c\|. \end{aligned}$$

Active phase : In the active phase the equation (2.22) follows the same stability analysis as explained in the previous phases. Now, consider the equation (2.21), the closed loop equation is given by:

$$\dot{e}_{vf} \in -k_f J_c^{-\top} K[V_f(e_{vf})] + J_c^{-\top} K[\widetilde{N}_c] \tag{3.15}$$

The derivative of the Lyapunov function is

$$\dot{V}_f = \xi_f^\top \dot{e}_{vf} \tag{3.16}$$

Substituting the closed loop equations (3.15) into (3.11), we have

$$\dot{V}_f = -k_f\xi_f^\top J_c^{-\top}\beta_f + \xi_f^\top J_c^{-\top}\zeta_f \tag{3.17}$$

Equation (3.17) is true for all $\xi_f \in \partial V_f$, for some $\beta_f \in \partial V_f$, and for some $\zeta_f \in K[\widetilde{N}_c]$. Since (3.17) is true for all $\xi_f \in \partial V_f$, we choose it to be $\xi_f = \arg\min\{\|\nu\| \mid \nu \in \partial V_f\}$ and from the convexity of the set ∂V_f, we obtain

$$\dot{V}_f \leq -k_f\xi_f^\top J_c^{-\top}\xi_f + \xi_f^\top J_c^{-\top}\zeta_f \tag{3.18}$$

Theorem 3.3 *With the choice of the control law as given in (3.8) and with the Lyapunov function (3.3), e_{vf} and e_f converge to zero, if k_f is chosen to be,*

$$k_f = \alpha_f + \eta_{ac} \tag{3.19}$$

where $\alpha_f > 0$, and $\eta_{ac} = \sup\{\|\eta_c\| : \eta_c \in K[\widetilde{N}_c]$.

The proof of this theorem is given in appendix II.

4 Discussions

We considered the problem of control of mechanical systems subject to unilateral constraints. This formulation convers a broad class of control problems for mechanical systems interacting with environment including collision, impacts etc. The mechanical system in motion is typically in one of the three phases of motion i.e. inactive, transition and active. We have formulated control laws for each phase of motion and shown stability for each phase.

We have not considered some of the practical issues that arise in implementing such a design on mechanical systems. Issues like, how one selects the duration of each phase? This directly influences the error bound at the end of that phase. An important issue is to show that the transition phase ends in finite time, resulting in the start of the active phase. The analysis done in the paper for the transition phase does not show that the transition phase is finite time. Presently, we do not know if there exists a rigorous mathematical proof to show that the impulses exactly die down in finite time, which means the transition phase is finite. Also during transition phase the mechanical systems goes through several impulses. During the transition phase the choice of time interval T_i over which the impulsive disturbances can be rejected is an other issue. In the future, we would like to analyze these situations in more detail. To illustrate the Hamiltonian formulation of the theory, we derive the equations for the UCB-NSK two link robot arm in appendix III.

5 Conclusions

We formulated a control design procedure for mechanical systems subject to unilateral constraints. The discontinuous differential equations describing such systems are derived using the Hamiltonian framework. The differential equations are transformed using a nonlinear transformation, two obtain two sets of differential equations, wherein only one set contains the contact force terms. Three phases of motion are defined for the system depending on the activation/deactivation of the constraints. The dynamics in each phase is represented by a set of differential equations. The impact forces are modelled as impulsive disturbances. Discontinuous control laws are proposed for the three phases. Stability of the closed loop system is shown by using Filippov's differential inclusion approach for differential equations with discontinuous righthand sides.

A Appendix

Proof of Theorem 3.1: The proof is given for equation (3.11), and the proof for (3.12) follows exactly the same lines. The derivative of the Lyapunov function V_c is

$$\dot{V}_c(e_{vc}) = \xi_c^\top \dot{e}_{vc}$$

The above equation is true $\forall \xi_c \in \partial V_c$. Substituting equation (3.11) into the above equation,

$$\begin{aligned}\dot{V}_c(e_{vc}) = &-k_{ic}\xi_c^\top \zeta_c + \xi_c^\top [\beta_c + Q_c\Lambda_c\dot{e}_c \\ &+ Q_{c1}\Lambda_u\dot{e}_u + \dot{Q}_c\Lambda_c e_c + \dot{Q}_{c1}\Lambda_u e_u - \dot{\theta}_{cd}]\end{aligned}$$

This equation is true $\forall \xi_c \in \partial V_c(e_{vc})$, for some $\zeta_c \in \partial V_c(e_{vc})$, for some $\beta_c \in K[\widetilde{N}_c]$. Choosing $\xi_c = \arg\min\{\|\nu\| \mid \nu \in \partial V_c\}$ and from the convexity of the set ∂V_c, we obtain

$$\begin{aligned}\dot{V}_c(e_{vc}) \leq &-k_{ic}\xi_c^\top \xi_c + \xi_c^\top [\beta_c + Q_c\Lambda_c\dot{e}_c + Q_{c1}\Lambda_u\dot{e}_u \\ &+ \dot{Q}_c\Lambda_c e_c + \dot{Q}_{c1}\Lambda_u e_u - \dot{\theta}_{cd}]\end{aligned} \quad \text{(A.1)}$$

Now, using the value of k_{ic} as given in theorem (3.1), we obtain

$$\begin{aligned}\dot{V}_c \leq &-\alpha_{ic}\|\xi_c\|^2 + (\|\xi_c\| - \|\xi_c\|^2)\|\|\eta_{ic} + Q_c\Lambda_c\dot{e}_c \\ &+ Q_{c1}\Lambda_u\dot{e}_u + \dot{Q}_c\Lambda_c e_c + \dot{Q}_{c1}\Lambda_u e_u - \dot{\theta}_{cd}\|\end{aligned} \quad \text{(A.2)}$$

The generalized gradient of $V_c(e_{vc})$ at 0 is $\partial V_c(0) = [-1,1]^m$ which is the unit m-cube. Since $V_c(e_{vc})$ is convex, the set $\partial V_c(e_{vc}) \cap (-1,1)^m$ is empty $\forall e_{vc} \neq 0$. Recall that the value of ξ_c is $\xi_c = \arg\min\{\|\nu_c\| \;:\; \nu_c \in \partial V_c(e_{vc})\}$, which implies $\|\xi_c\| \geq 1 \; \forall e_{vc} \neq 0$. Therefore we obtain,

$$\dot{V} \leq -\alpha_{ic}\|\xi_c\|^2 \quad \text{(A.3)}$$

So we have $V_c(e_{vc}(t))$ absolutely continuous and $\dot{V}_c$ is negative, therefore from the Lyapunov theorem, $e_{vc} \to 0$. Proceeding along the same lines for the unconstrained directions it can be shown that $e_{vu} \to 0$. Now, from the definition of errors, we can write

$$e_v := \begin{bmatrix} e_{vc} \\ e_{vu} \end{bmatrix} = \begin{bmatrix} (\theta_c - \theta_{cd}) + Q_c(q)\Lambda_c e_c + Q_{c1}(q)\Lambda_u e_u \\ (\theta_u - \theta_{ud}) + Q_u(q)\Lambda_u e_u + Q_{c2}(q)\Lambda_c e_c \end{bmatrix}$$

After reorganizing and recalling the decomposition of $G(q)M(q)$ we obtain $e_v = (\theta - \theta_d) + G(q)M(q)\Lambda e$. Premultiplying e_v by $M^{-1}(q)G^{-1}(q)$ we obtain, $\dot{e} + \Lambda e \to 0$. Thus, $\dot{e} \to 0$ and $e \to 0$. ◇

B Appendix

Proof of Theorem 3.3: Substituting the value of k_f given in theorem (3.3), in the derivative of the Lyapunov function (3.18), we get

$$\dot{V}_f \leq -\alpha_f\|\xi_f\|^2 + (\|\xi_f\| - \|\xi_f\|^2)\|\eta_{ac}\| \quad \text{(B.1)}$$

Using, similar arguments as given in appendix I, we can see that, $e_{vf} \to 0$, and from the error equation (3.16), e_f is bounded, and hence $e_f \to 0$. ◇

C Appendix

Example : Here, we give an illustrative example for the theory developed in the paper. The mechanical system under consideration is a two link UCB-NSK planar robot arm shown in Fig. 1. The mass matrix for the robot is

$$M(q) = \begin{bmatrix} a_1 + 2a_3c_2 & a_2 + a_3c_2 \\ a_2 + a_3c_2 & a_2 \end{bmatrix}$$

where $c_i = \cos(q_i), c_{ij} = \cos(q_i + q_j)$. The constraint Φ is given by $\Phi(q) := l_1c_1 + l_2c_{12} - d$. The constrained Lagrangian for the

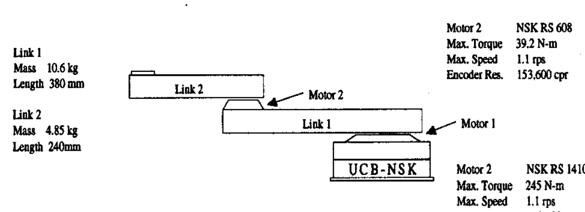

Figure 1: UCB-NSK Two Link Robot

system is

$$\mathcal{L}^* = \frac{1}{2}\dot{q}^\top M(q)\dot{q} + \lambda_c\Phi$$

and the generalized momenta are

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} (a_1 + 2a_3c_2)\dot{q}_1 + (a_2 + a_3c_2)\dot{q}_2 \\ (a_2 + a_3c_2)\dot{q}_1 + a_2\dot{q}_2 \end{bmatrix}$$

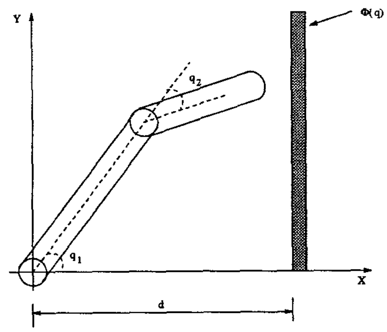

Figure 2: Robot and the Constraint Surface

We compute the inverse of the mass matrix before giving the expression for the Hamiltonian, which is

$$M^{-1}(q) = \frac{1}{\Delta_m}\begin{bmatrix} a_2 & -a_2 - a_3c_2 \\ -a_2 - a_3c_2 & a_1 + 2a_3c_2 \end{bmatrix}$$

where $\Delta_m = a_1a_2 - a_2^2 - a_3^2c_2^2$ is the determinant of the mass matrix. The Hamiltonian of the system is

$$\mathcal{H} = \frac{1}{2\Delta_m}\{a_2p_1^2 + (a_1 + 2a_3c_2)p_2^2 - 2(a_2 + a_3c_2)p_1p_2.\}$$

The Hamilton's equations of motion are

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \frac{1}{\Delta_m}\begin{bmatrix} a_2p_1 - (a_2 + 2a_3c_2)p_2 \\ -(a_2 + a_3c_2)p_1 + (a_1 + 2a_3c_2)p_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2\Delta_m^2}(p_1p_2y_1 + p_2^2y_2) \end{bmatrix} + \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} + \begin{bmatrix} -l_1s_1 - l_2s_{12} \\ -l_2s_{12} \end{bmatrix}\lambda_c$$

where

$$y_1 = \frac{(a_1a_2 - a_2^2 + 2a_2a_3c_2 + a_3^2c_2)a_3s_2)}{\Delta_m^2}$$

$$y_2 = \frac{(a_2^2 - a_1a_2 - a_3^2c_2^2 - a_1a_3c_2)2a_3s_2}{\Delta_m^2}$$

Now, the matrix $G(q)$ can be constructed as

$$G(q) := \begin{bmatrix} G_c(q) \\ G_u(q) \end{bmatrix} = \begin{bmatrix} -l_1c_1 - l_2c_{12} & -l_2c_{12} \\ -l_2c_{12} & l_1c_1 + l_2c_{12} \end{bmatrix}$$

and its inverse is given by

$$G^{-1}(q) = \frac{1}{\Delta_g}\begin{bmatrix} l_1c_1 + l_2c_{12} & l_2c_{12} \\ l_2c_{12} & -l_1c_1 - l_2c_{12} \end{bmatrix}$$

where $\Delta_g = -(l_1c_1 + l_2c_{12})^2 - l_2^2c_{12}^2$ is the determinant of the matrix $G(q)$. Now consider the transformation of p by $G(q)$ as $\theta = G(q)p$, the equations of motion in the transformed coordinates are

$$\dot{q} = M^{-1}(q)G^{-1}(q)\theta$$

$$\dot{\theta} = -\frac{1}{2}G(q)(I_n \otimes \theta^\top G^{-\top}(q))\frac{\partial M^{-1}}{\partial q}G^{-1}(q)\theta + \dot{G}G^{-1}(q)\theta + G(q)\tau + G(q)G_c^\top(q)\lambda_c.$$

We compute the different matrices involved and express θ in terms of θ_c and θ_u.

$$(I_n \otimes \theta^\top G^{-\top})\frac{\partial M^{-1}}{\partial q} = \begin{bmatrix} 0 & z_2y_1 \\ 0 & z_1y_1 + z_2y_2 \end{bmatrix}$$

where $z_1 = (l_1c_1 + l_2c_{12})\theta_c + l_2c_{12}\theta_u$, and $z_2 = l_2c_{12}\theta_c - (l_1c_1 + l_2c_{12})\theta_u$ and

$$G(I_n \otimes \theta^\top G^{-\top})\frac{\partial M^{-1}}{\partial q}G^{-1} = \frac{1}{\Delta_g}\begin{bmatrix} l_2c_{12}w_1 & -(l_1c_1 + l_2c_{12})w_1 \\ l_2c_{12}w_2 & -(l_1c_1 + l_2c_{12})w_2 \end{bmatrix}$$

where $w_1 = -z_2y_1(l_1c_1 + l_2c_{12}) - l_2c_{12}(z_1y_1 + z_2y_2)$, and $w_2 = -l_2c_{12}z_2y_1 + (l_1c_1 + l_2c_{12})(z_1y_1 + z_2y_2)$. Now, for simplicity define

$$\dot{G}(q,\theta)G^{-1}(q) = \begin{bmatrix} g_{11}(q,\theta) & g_{12}(q,\theta) \\ g_{21}(q,\theta) & g_{22}(q,\theta) \end{bmatrix}$$

Notice that $\dot{G}$ depends on both q and θ. So the equations of motions can be written as

$$\begin{aligned} \dot{q} &= M^{-1}(q)G^{-1}(q)\theta =: \begin{bmatrix} b_{11}(q) & b_{12}(q) \\ b_{21}(q) & b_{22}(q) \end{bmatrix}\begin{bmatrix} \theta_c \\ \theta_u \end{bmatrix} \\ \dot{\theta}_c &= -N_c(q,\theta) + \tau_c + G_cG_c^\top\lambda_c \\ \dot{\theta}_u &= -N_u(q,\theta) + \tau_u \end{aligned}$$

where $N_c(q,\theta) = \frac{1}{\Delta_g}(l_2c_{12}w_1\theta_c - (l_1c_1 + l_2c_{12})w_1\theta_u) + g_{11}\theta_c + g_{12}\theta_u$ and $N_u(q,\theta) = \frac{1}{\Delta_g}(l_2c_{12}w_2\theta_c - (l_1c_1 + l_2c_{12})w_2\theta_u) + g_{21}\theta_c + g_{22}\theta_u$. The equations of motion are four first order differential equations in q and θ,

$$\dot{q}_1 = b_{11}(q)\theta_c + b_{12}(q)\theta_u \qquad (C.1)$$

$$\dot{q}_2 = b_{21}(q)\theta_c + b_{22}(q)\theta_u \qquad (C.2)$$

$$\dot{\theta}_c = -N_c + \tau_c + ((l_1c_1 + l_2c_{12})^2 + (l_2c_{12})^2)\lambda_c \qquad (C.3)$$

$$\dot{\theta}_u = -N_u + \tau_u \qquad (C.4)$$

Now from (C.1) to (C.4), we can obtain the equations for different phases and design the control laws for each phase as given in section 3.

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