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Uncontrollable Singularities in Nonlinear Systems

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Abstract— **The problem of tracking through singular submanifolds is extended to include regions of uncontrollability. A special, pragmatic class of nonlinear systems with uncontrollable, singular points is analyzed. The unique behavior of these systems is compared to systems with *controllable*, input singularities such as the well-known ball and beam system that has enjoyed considerable attention throughout the literature. Analysis reveals the deficiency of existing control solutions to properly handle an input singularity that is also uncontrollable. A sufficient condition for successfully maneuvering through an uncontrollable, singular submanifold is derived providing a basis for reference trajectory design. An illustrative example of practical applicability is presented for a disk drive, commutational ramp load actuator.**

I. Introduction

An input singularity occurs when the input vector field of a system in normal form becomes null. Therefore, from an input-output perspective, the system relative degree is undefined. Previous research investigating system characteristics at or near input singularities primarily propose system approximation methods. In [1], a method is developed for reference trajectory tracking of systems through singular points. A control law is designed based on an approximation of the true system by neglecting higher order terms resulting in a definitive input/output map. Bounded tracking is only guaranteed in a neighborhood of the true system singular points because the neglected terms may become significant further away from the singular condition. System characteristics are defined and presented as necessary conditions for stability of the error dynamics under the proposed control law. The ball and beam system (BBS) exhibits an ill-defined relative degree and was used extensively as a demonstrative platform for studying this phenomena. The work of [1] was refined in [2] where the authors used exact input-output tracking away from the singular condition and switched to the approximate law of [1] within some threshold. The switching scheme included an exponential term to ensure a finite switching rate. The control law required the necessary conditions of [1] in addition to a “slowly varying” reference trajectory near singular points and stable zero dynamics. The BBS was used for controller evaluation. A similar switching control scheme (without the exponential term) was introduced in [3] and extensive analysis was performed to determine what effect the zero dynamics would have on the control performance. Sufficient conditions were given for bounded tracking and results were presented for the BBS. Approximate tracking through the input singularity requires the system to satisfy local controllability within the region of operation.

This work introduces a special class of nonlinear systems which become uncontrollable at points of singularity. The practical significance of the problem focuses on reducing cost or expanding the operating range of a physical system. For example, a region of uncontrollable, singularity could exist naturally for an aerospace vehicle as a result of design constraints, actuator malfunction, or damage. A strategy to maneuver through this region could expand or reinstate desirable limits on the flight envelope. As another example, consider that disk drive ramp load actuators are designed with additional magnet and coil material for the sole purpose of preventing the existence of an uncontrollable, input singularity during a ramp load/unload maneuver. A controller capable of steering through the singular region would allow the use of a lower cost disk drive actuator while maintaining the desired seek performance capability. A sufficient condition is derived to facilitate a maneuver through an uncontrollable, singular submanifold. As an example of practical applicability, the analysis is developed around a disk drive, commutational ramp load (CRL) actuator shown in Figure 1. Considerations for a potential trajectory design and control strategy for the CRL actuator example are discussed.

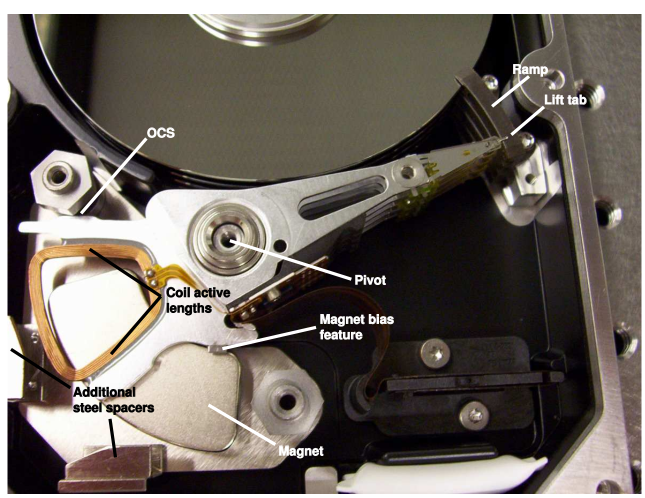

Fig. 1. Commutational ramp load disk drive.

II. THE BALL AND BEAM SYSTEM REVISITED

Consider the BBS of [1] defined on manifold M without the influence of gravity ($G = 0$). For the set of local physical coordinates $[r\ \dot{r}\ \theta\ \dot{\theta}]^T = [x_1\ x_2\ x_3\ x_4]^T$ and output $y = x_1$, the system becomes

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 x_4^2 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= u \end{aligned} \tag{1}$$

and continues to exhibit an undefined relative degree on $M_s^{BB} \in M$ where $M_s^{BB} := \{x \in \mathbb{R}^4 \mid x_1 x_4 = 0\}$. Following the system approximation construction procedure gives

$$\begin{aligned} \dot{\xi}_1 &= \underbrace{x_2}_{\xi_2=\alpha_2(x)} \\ \dot{\xi}_2 &= \underbrace{x_1 x_4^2}_{\xi_3=\alpha_3(x)} \\ \dot{\xi}_3 &= \underbrace{x_2 x_4^2}_{\xi_4=\alpha_4(x)} + \underbrace{2x_1 x_4\, u}_{\beta_3(x,u)} \\ \dot{\xi}_4 &= \underbrace{x_1 x_4^3}_{\xi_4=\alpha_4(x)} + \underbrace{2x_2 x_4\, u}_{\beta_4(x,u)} \\ &\vdots \\ \dot{\xi}_k &= \alpha_{k+1}(x_1, x_2, x_4) + \beta_k(x_1, x_2, x_4)u \end{aligned} \tag{2}$$

where continued Lie derivatives reveal that all terms with u are multiplicative in x_4. Therefore, without gravity, the BBS fails to possess a *robust relative degree* [1] and the position of the ball can only be controlled through the centripetal acceleration term. Furthermore, the absence of gravity alters controllability at the singular point. Evaluation of the system controllability at $x_c = [x_1\ x_2\ x_3\ x_4]^T = [0\ 0\ 0\ 0]^T$ with the new dynamics of (2) gives the controllability subalgebra

$$\mathcal{C}(x_c) = [ad_f^0 g \mid ad_f g] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad ad_f^k g(x_c) = [\mathbf{0}],$$

for all $k > 1$. So the controllability distribution $\Delta_{\mathcal{C}}(x_c) = \text{span}\{\mathcal{C}(x_c)\}$, has dimension $\dim \Delta_{\mathcal{C}}(x_c) = 2 < 4$, and the system is, therefore, not locally reachable from x_c on M. From a physical perspective, radial motion $(r, \dot{r})$ of the ball cannot be induced by a nonzero angular position of the beam. Without gravity, there is no reachable state in ball position or velocity from x_c resulting from an applied torque to the beam. Values of beam angular position and velocity remain reachable, however, implying the existence of a foliation parameterized by all admissible values of beam angular position, p_3. Therefore, a non-empty reachable set, $\mathcal{R}(x_c) \subset S_{x_c}^{BB}$, exists where the submanifold

$$S_{x_c}^{BB} := \{x \in M \mid x_1 = 0, x_3 = p_3\}$$

constitutes a leaf of the foliation for each fixed constant, p_3.

III. DISK DRIVE RAMP LOAD ACTUATOR

Disk drive read/write heads are very sensitive to external shocks. A severe shock, such as that resulting from a drop, forces the read/write heads to impact the disks causing catastrophic damage to drive functionality. To combat the issue, many disk drive manufacturers prefer to keep the heads parked completely off the disks when the drive is not operating. The common method adopted in industry is to park the tip of the actuator on a ramp located outside the disk outer diameter. Figure 1 shows a typical non-operational configuration with actuator parked on a ramp and heads suspended off the disks. When the drive is powered on, the actuator moves the heads off the ramp and onto the disks at a regulated velocity where normal operation occurs. When the drive is powered off, residual energy from the disk spindle motor back-emf is used to move the actuator, unloading the heads off the disks and onto the ramp. A disk drive with ramp load capability typically possesses increased magnet and coil volume resulting from the additional angle required to park the heads off the disk. Additional cost and seek performance reduction is the resulting consequence. However, a *commutational* ramp load actuator (CRL) can be incorporated to eliminate extra costs and maintain performance while preserving the shock resistance benefits of ramp loading for disk drives [4]. Unfortunately, the highly nonlinear actuator dynamics possess an uncontrollable, input singularity that cannot be avoided during a ramp load maneuver. A successful ramp load maneuver constitutes moving the actuator through the singularity and equilibrium points and launching off the ramp at some predetermined regulated velocity.

A. Modeling

The actuator can be described by a combination of mechanical and electrical dynamic equations. The mechanical dynamics are represented by

$$J\ddot{\theta} = K_t(\theta)i - T_b(\theta) - T_f\ \text{sat}(\dot{\theta}) \tag{3}$$

where θ is the actuator arm angular position, J is the actuator inertia, and i is the current applied in the coil. The torque factor, K_t, which is relatively constant during operation in the data zone, becomes a function of the actuator arm angle while on the ramp. For the disk drive depicted in Figure 1, there exists a condition rendering the actuator uncontrollable by induced current while traveling on the ramp. Figure 2 illustrates the torque factor characteristics along the ramp angle for the disk drive of Figure 1. The uncontrollable condition occurs at a critical angle, θ_c, corresponding to $K_t(\theta_c) = 0$. The critical angle is measured as the actuator rotates counter-clockwise from the parked position at the outer crash stop (OCS) (Fig 1). The ramp angle, θ_r, is the total angle the actuator travels from the OCS to the end of the ramp. Also shown in Figure 1 is a feature that induces a magnetic bias torque on the actuator arm. The bias torque, $T_b(\theta)$, is a function of actuator arm angle and is required to prevent an unrecoverable condition at θ_c that may be imparted by an external disturbance. The bias assists the actuator in returning to the OCS without requiring current to be present in the coil. The dynamic friction torque $T_f\,\mathrm{sat}(\dot{\theta})$,

$$\mathrm{sat}(\dot{\theta}) = \begin{cases} -1, & \dot{\theta} < -\omega_l \\ \dfrac{\dot{\theta}}{\omega_l}, & -\omega_l \le \dot{\theta} \le \omega_l, \\ 1, & \dot{\theta} > \omega_l \end{cases} \tag{4}$$

results from the suspension lift tab/ramp surface interaction during a L/UL operation and is dependent on the direction of actuator motion. The saturation component is a function of the desired final ramp loading velocity, ω_l.

Equation (5) represents the electrical dynamics where R and L are the coil resistance and inductance, respectively.

$$V_s = Ri + L\frac{di}{dt} + K_t(\theta)\dot{\theta} \tag{5}$$

The supply voltage, V_s, is available for controlling the system. Referring to Figure 2 and noting that

$$K_t(\theta) \begin{cases} < 0, & \theta < \theta_c \\ = 0, & \theta = \theta_c \\ > 0, & \theta > \theta_c \end{cases} \tag{6}$$

the commutation requirement becomes apparent by rearranging (3) and (5) as

$$\frac{d^2\theta}{dt^2} = \frac{1}{J}\left[K_t(\theta)i - T_b(\theta) - T_f\,\mathrm{sat}(\dot{\theta})\right] \tag{7}$$

$$\frac{di}{dt} = \frac{1}{L}\left[-K_t(\theta)\dot{\theta} - Ri + V_s\right] \tag{8}$$

When $\theta < \theta_c$, $K_t(\theta) < 0$. Therefore, from (7), $i < 0$ is required to overcome the bias and friction torques and move the actuator a positive angular displacement toward the end of the ramp. When $\theta > \theta_c$, $K_t(\theta) > 0$ and the current must change sign ($i > 0$) to maintain the direction of actuator motion.

There also exists two forced equilibrium points, θ_{eq-} and θ_{eq+}, depending on the location of the actuator arm relative to the critical angle, θ_c. Using equations (3) and (5) the equilibrium points are shown to occur when

$$K_t(\theta_{eq})V_{sat} - RT_b(\theta_{eq}) = 0 \tag{9}$$

where V_{sat} is the saturation voltage for the system. Knowing $K_t(\theta)$ and $T_b(\theta)$, the equilibrium points can be determined as a function of V_{sat} and the coil resistance where $\theta_{eq-} < \theta_c < \theta_{eq+}$. The goal is to maneuver the actuator from the OCS through the critical angle and equilibrium points and load onto the disks within the desired velocity constraint.

Choosing the states as $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = i$, and denoting the control input as $u = V_s$ and velocity output $y = x_2$, gives the third-order state space representation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \mu_2[K_t(x_1)x_3 - T_b(x_1) - T_f\,\mathrm{sat}(x_2)] \\ \dot{x}_3 &= \mu_3[-K_t(x_1)x_2 - Rx_3 + u] \\ y &= x_2 \end{aligned} \tag{10}$$

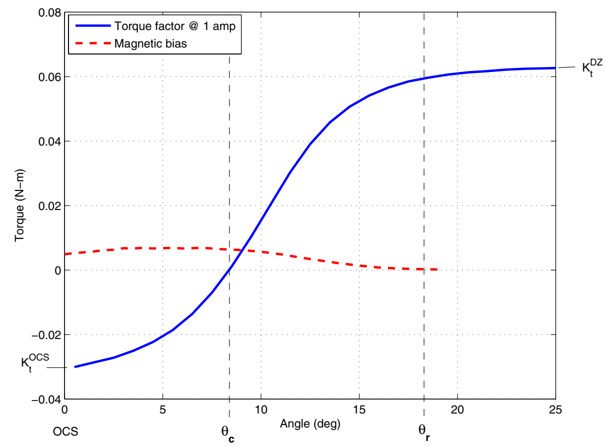

Fig. 2. Actuator torque profiles along the ramp angle, θ_r.

defined on a manifold, M where $\mu_2 = \frac{1}{J}$ and $\mu_3 = \frac{1}{L}$. The motor torque factor and bias can both be represented as polynomial expressions of the form

$$K_t(x_1) = \sum_{k=0}^{m} b_k x_1^k \qquad T_b(x_1) = \sum_{k=0}^{n} c_k x_1^k \tag{11}$$

where b_k and c_k are the k-th order coefficients.

B. Analysis

Since the bias and friction term are not multiplicative with x_3, for the purpose of analysis, consider $T_b(x_1) = T_f = 0$,for all $x \in M$ such that

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= K_t(x_1)x_3 \\ \dot{x}_3 &= -K_t(x_1)x_2 - x_3 + u \end{aligned} \tag{12}$$

where inertia, resistance, and inductance all take values of unity. The system can be written as

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{13}$$

where the drift and input vector fields take the form

$$f(x) = \begin{bmatrix} x_2 \\ K_t(x_1)x_3 \\ -K_t(x_1)x_2 - x_3 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

With output, $y = x_2$, the input/output relationship is

$$\begin{aligned} y &= x_2 \\ \dot{y} &= K_t(x_1)x_3 \\ \ddot{y} &= \underbrace{\frac{\partial K_t}{\partial x_1}x_2x_3 - K_t^2(x_1)x_2 - K_t(x_1)x_3}_{b(x)} + \underbrace{K_t(x_1)}_{a(x)}u \end{aligned}$$

If $K_t(x_1)$ is nonzero in the region of interest, then the system has relative degree $\gamma = 2$ and the control law

$$u = \frac{-b(x) + v}{a(x)} \tag{14}$$

can be used to provide the linear input/output relationship $\ddot{y} = v$. However, if there exists some

$$M_s^{RL} := \{x \in M \mid K_t(x_1) = 0\}$$

then a singular condition exists along with an undefined relative degree. An approximate system using the method of [1] with diffeomorphism $\Pi : x \mapsto (\xi, \eta)$ where $\xi_1 = \alpha_1(x) = x_2$ gives

$$\begin{aligned}
\dot{\xi}_1 &= \underbrace{K_t(x_1)x_3}_{\xi_2=\alpha_2(x)} \\
\dot{\xi}_2 &= \underbrace{\frac{\partial K_t}{\partial x_1}x_2x_3 - K_t^2(x_1)x_2 - K_t(x_1)x_3}_{\xi_3=\alpha_3(x)} + \underbrace{K_t(x_1)u}_{\beta_2(x)} \\
\dot{\xi}_3 &= K_t^2(x_1)x_2 - 3\frac{\partial K_t}{\partial x_1}K_t(x_1)x_2^2 + \left[\frac{\partial^2 K_t}{\partial x_1^2}x_2^2 \right. \qquad (15) \\
&\quad \left. + \frac{\partial K_t}{\partial x_1}(K_t(x_1)x_3 - 2x_2) - K_t^3(x_1) + K_t(x_1)\right] x_3 \\
&\quad - \left[\frac{\partial K_t}{\partial x_1}x_2 + K_t(x_1)\right] u \\
&\vdots \\
\dot{\xi}_k &= \alpha_{k+1}(K_t(x_1), x_2, x_3) + \beta_k(x, u) \\
\dot{\eta} &= x_1
\end{aligned}$$

Analysis of (15) reveals that all terms with x_3 and u are multiplicative in K_t and x_2 defining a singular submanifold

$$\left\{x \in M \;\middle|\; \frac{\partial K_t}{\partial x_1}x_2 = K_t(x_1)\right\}$$

Therefore, the CRL system fails to have a robust relative degree. It is of interest to determine the reachable set from the singular manifold. Even if exact input/output linearization cannot be achieved from M_s^{RL}, the potential of reaching other states may exist. Without loss of generality, assume for the *exact* system, $K_t(0) = 0$ and the equilibrium point $x_c^T = [0\ 0\ 0] \in M_s^{RL}$. Taking successive Lie brackets of f and g on M_s

$$ad_f^k g(x_c) = -\frac{\partial f}{\partial x}(x_c)\, ad_f^{(k-1)} g(x_c), \quad k = 0, 1, 2, \ldots$$

results in the controllability subalgebra at x_c

$$\mathcal{C}(x_c) = \begin{bmatrix} 0 & \cdots \\ 0 & \cdots \\ (-1)^k & \cdots \end{bmatrix}, \quad k = 0, 1, 2, \ldots$$

Therefore, the controllability distribution has dimension $\dim \Delta_{\mathcal{C}}(x_c) = 1$ and the system is not locally reachable at x_c. Furthermore, there exists no output resulting in a relative degree $\gamma = 3$ system on x_c. However, the reachable set from x_c, $\mathcal{R}(x_c)$ is non-empty suggesting that some portion of the system can continue to be manipulated with control input leading to the following lemma.

Lemma 1: Consider the CRL system with dynamics (12) and controllability distribution $\Delta_{\mathcal{C}}(x_c)$ where $\dim \Delta_{\mathcal{C}}(x_c) = 1 < 3$. There exists a neighborhood U of x_c for each constant p_2 such that the submanifold

$$S_{x_c}^{RL} := \{x \in M_s \mid x_1 = 0, x_2 = p_2\} \qquad (16)$$

is an integral manifold of $\Delta_{\mathcal{C}}(x_c)$ Then for any neighborhood U of x_c and for all $T > 0$, $\mathcal{R}(x_c)$ is contained in $S_{x_c}^{RL}$. Furthermore, $\mathcal{R}(x_c)$ contains a non-empty open set of the integral manifold $S_{x_c}^{RL}$. Hence the system restricted to $S_{x_c}^{RL}$ is locally reachable. This result follows from the general result in [5].

The submanifold $S_{x_c}^{RL}$ is a leaf of the foliation generated by parameterizing all admissible values of angular velocity according to $x_2 = p_2$ for each constant p_2. Lemma 1 can be applied to each foliation leaf resulting in a velocity independent reachable set. The reachable set in this case is the set of all admissible values of current with dynamics described by

$$\dot{x}_3 = -K_t(x_1)x_2 - x_3 + u \qquad (17)$$

and the point $x_c \in M_c^{RL}$ is defined as an uncontrollable, input singularity or a *critical point* where

$$M_c^{RL} := \{x \in M_s^{RL} \mid \dim \Delta_{\mathcal{C}}(x_c) < n\}$$

Introduction of a bias term in the CRL system alters the drift vector field. For simplicity, it is assumed that the constant and first-order coefficient terms of the bias polynomial estimate are $c_0 = c_1 = 1$. The drift vector field at $x_s^T = [0\ 0\ 0]$ now becomes

$$f(x_s) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \qquad \frac{\partial f}{\partial x}(x_s) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

It can be verified that the controllability subalgebra remains unchanged and, therefore, the bias offers no assistance in defining a relative degree. However, the bias does change the equilibrium state of the system.

Following from Section III-A, there exists two *forced* equilibrium points, x_{eq-} and x_{eq+}, depending on the location of the actuator arm relative to the critical angle, x_{1c}.

$$K_t(x_{1eq})u_{sat} - RT_b(x_{1eq}) = 0, \quad x_{1eq-} < x_{1c} < x_{1eq+}$$

where u_{sat} is the saturation voltage for the system. These equilibria occur as a result of the magnetic bias neutralizing the torque generated for a given value of current. Consider $U_c \subset M$ containing M_c^{RL} such that

$$U_c := \{x \in M \mid x_{1eq-} < x_1 < x_{1eq+}\}$$

For a given a set of disk drive physical parameter values, four possibilities exist corresponding to the dynamics of equation (10) when maneuvering through U_c with and without input commutation.

Case 1: No commutation ($u = -u_{sat}$) The actuator initially moves away from the OCS through x_{1c}, but is forced back toward the OCS when $x_1 > x_{1c}$. The cycle continues and a stable focus results about x_{1eq-} as shown in Figure 3. The remaining three scenarios occur when the input voltage switches polarity

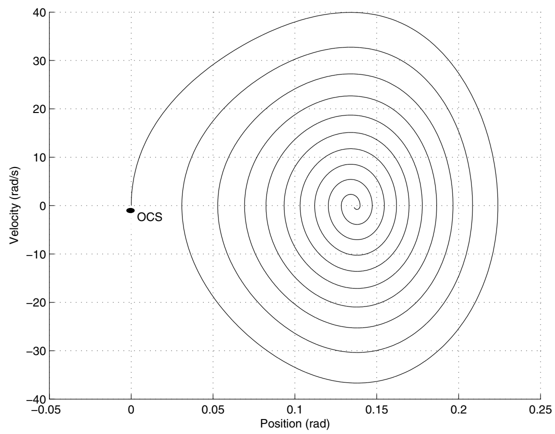

Fig. 3. Actuator ramp dynamic characteristics without commutation

$u = -u_{sat},\ 0 \le t < t_s$ or $u = u_{sat},\ t \ge t_s$ at some switching time, t_s.

Case 2: The actuator initially moves away from the OCS. The input voltage reverses polarity at $t_s = 3.5$ msec at an angle $x_1(t_s) < x_{1eq+}$. There is not enough initial velocity to reach x_{1eq+} and the actuator returns to the OCS (Fig. 4).

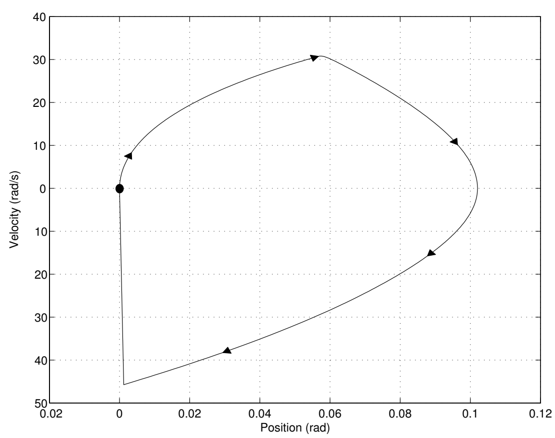

Fig. 4. Characteristics with commutation. Actuator forced back to OCS

Case 3: Similar to Case 2, but $t_s = 3.929$ msec occurs later than Case 2 where there is just enough initial velocity such that $x_2 = 0$ at x_{1eq+}. As shown in Figure 5, the actuator rests at x_{1eq+}.

Case 4: The input polarity reversal occurs at $t_s = 6.0$ msec such that $x_2 > 0$ at x_{1eq+} and the actuator moves through x_{1eq+} and loads the heads onto the disk.

The latter case is desired and suggests that there exists a minimum, critical angular velocity, x_{2c}, such that if the actuator reaches x_{2c} prior to reaching, U_c, it can successfully pass through U_c to some $\{x \in M \mid x_1 > x_{1eq+}\}$. Transform the CRL system to a new set of local physical coordinates

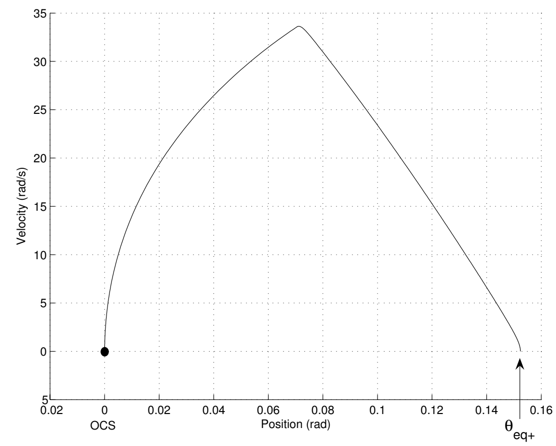

Fig. 5. Characteristics with commutation. Actuator rests at x_{1eq+}

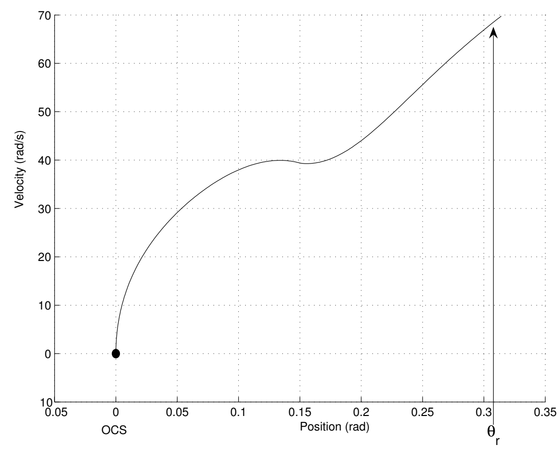

Fig. 6. Characteristics with commutation. Actuator loads onto disks

$z = [\theta\ \omega\ \alpha]$ on M through the diffeomorphism $\Pi : x \mapsto z$

$$\begin{aligned} \theta &= x_1 \\ \omega &= x_2 \\ \alpha &= \mu_2[K_t(x_1)x_3 - T_b(x_1) - T_f] \end{aligned} \tag{18}$$

where θ, ω, α are the angular position, velocity, and acceleration, respectively, defined as

$$\omega = \frac{d\theta}{dt}, \quad \alpha = \frac{d\omega}{dt} \tag{19}$$

The equations in (19) can be rewritten to establish a time independent relationship between angular velocity and acceleration

$$\omega\frac{d\omega}{d\theta} = \alpha(\theta) \quad \text{or} \quad \omega\, d\omega = \alpha(\theta)\, d\theta \tag{20}$$

Integrating both sides of (20) and noting the expression for angular acceleration from (18) gives

$$\Delta\omega^2 = 2\mu_2 \left\{ \int_{\theta_o}^{\theta_{eq+}} [K_t(\zeta)i - T_b(\zeta) - T_f]\, d\zeta \right\}$$

where $\Delta\omega^2 = \omega^2(\theta_{eq+}, i) - \omega_c^2(\theta_o, i)$ and $\omega_c(\theta_o, i)$ is the actuator initial velocity at some $\theta_o < \theta_{eq+}$. Let $T_{fc} = T_f(\theta_{eq+} - \theta_o)$. If

$$\omega_c^2(\theta_o, i) = \left| 2\mu_2 \left\{ \int_{\theta_o}^{\theta_{eq+}} [K_t(\zeta)i - T_b(\zeta)]\, d\zeta - T_{fc} \right\} \right|,$$

for all $z \in U_c$, then $\omega^2(\theta_{eq+}, i) \geq 0$ and the actuator will either pass through or rest at θ_{eq+}, respectively. If

$$\omega_c^2(\theta_o, i) > \left| 2\mu_2 \left\{ \int_{\theta_o}^{\theta_{eq+}} [K_t(\zeta)i - T_b(\zeta)]\, d\zeta - T_{fc} \right\} \right|, \quad (21)$$

for all $z \in U_c$, then $\omega^2(\theta_{eq+}, i) > 0$ and the actuator will pass through θ_{eq+} for all $z \in U_c$. Therefore an upper bound on the right side of (21) can be determined resulting in an input that guarantees (21) is satisfied. The above example for the CRL actuator leads to a general formulation for a class of systems defined on uncontrollable, singular manifolds.

Theorem 1: Consider the nonlinear system (13) defined on a smooth, connected, n-dimensional manifold M represented in the normal form as

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\vdots \\ \dot{\xi}_{k-1} &= \xi_k \\ \dot{\xi}_k &= \xi_{k+1} \\ \dot{\xi}_{k+1} &= \rho(\xi) + \beta(\xi)u \\ \dot{\eta} &= q(\xi, \eta) \end{aligned} \quad (22)$$

where $2 \leq k \leq (n-1)$. Let U_c be an open neighborhood of a critical point $\xi_c \in M_c$ where

$$M_c := \{\xi \in M \mid \beta(\xi) = 0, \xi_j = 0,\, j = 2, \ldots, k\}$$

and consider the one-dimensional submanifold $\Theta_c \subset U_c$,

$$\Theta_c := \{\xi \in U_c \mid \xi_{k-1} \neq 0,\, \xi_{k-1} \in (a, b)\}$$

where $[a, b]$ is in the closure of U_c. There exists a set

$$M_\Omega := \left\{ \xi \in M \;\middle|\; \xi_k^2 > \left| 2\int_a^b \xi_{k+1} d\xi_{k-1} \right|,\ \forall \xi \in U_c \right\}$$

such that if the system is driven to M_Ω, it can successfully pass through U_c.

Proof: Integration of the time-independent relationship

$$\xi_k d\xi_k = \xi_{k+1} d\xi_{k-1}$$

results in

$$\xi_{kb}^2 - \xi_{ka}^2 = 2\int_{\Theta_c} \xi_{k+1} d\xi_{k-1} = 2\int_a^b \xi_{k+1} d\xi_{k-1}$$

where ξ_{ka} and ξ_{kb} are the values of ξ_k in the submanifold, Θ_c, corresponding to $\xi_{k-1} = a$ and $\xi_{k-1} = b$, respectively. Therefore, if

$$\xi_{ka}^2 > \left| 2\int_a^b \xi_{k+1} d\xi_{k-1} \right|, \quad \forall \xi \in U_c$$

then $\xi_{kb}^2 > 0$ and $\dot{\xi}_{k-1} > 0$ for $\xi_{k-1} = b$. Since b is in the boundary set of U_c, the system moves to a point outside of U_c. ■

The sufficient condition of Theorem 1 can be used as a basis for generation of an admissible reference trajectory that can be tracked using closed loop control to maneuver a system through an uncontrollable, input singularity. A detailed trajectory generation example based on Theorem 1 is given in [6].

C. Control Strategy

A tracking controller must be synthesized to complement the preceding trajectory design. As suggested in the previous section, the two are not mutually exclusive and, therefore, control law selection should be considered prior to developing a reference trajectory. The CRL actuator case, in particular, requires special attention to gain selection near the critical region. For example, gains inversely proportional to torque factor could result in saturation issues. Compensation for the input polarity reversal requirement must exist in any reference model implementation as indicated by (6)-(8). Controller flexibility is degraded further by the fact that full state feedback is not available. A control law must be based on current measurement only with or without position and velocity observation. An example control synthesis for the CRL actuator can be found in [4].

IV. Conclusion

The paper discusses a unique class of nonlinear systems defined on uncontrollable, singular submanifolds. Previous control methodologies designed to contend with singular regions are not effective when a point of uncontrollability is coincident. It was determined that trajectory and controller design are not mutually exlusive and a sufficient condition for manuevering through the uncontrollable singularity is given with application to a disk drive commutaional ramp load actuator. A general formulation is developed for the class of systems possessing this quality.

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