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Adaptive controller and observer design for a class of nonlinear systems*

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Abstract: Design of a stable adaptive controller and observer for a class of nonlinear systems is considered. Stable adaptive observer designs in the existing literature are generally based on the assumption that the nonlinearities in the system dynamics are functions of measured variables and inputs. In this work, a broader class of nonlinear systems that contain the product of an unmeasurable state and an unknown parameter are considered. The nonlinear system is transformed into a suitable form which allows for the design of a stable adaptive controller and a stable nonlinear observer using a parameter dependent Lyapunov function. The design process is shown on a simple example and then extended to the general case. Simulation results on two distinct examples are shown and discussed for the proposed scheme.

1 Introduction

Many practical applications require estimation of the states and parameters that can be used in designing a stable control algorithm; the unmeasurable states and parameters are generally estimated based on the knowledge of the physical system, such as a model, and the available measurements. Design of a stable adaptive observer that simultaneously estimates the unmeasurable state and the unknown parameters for a general class of nonlinear systems is still an open problem. This has led to continued strong interest over the years in the development of stable adaptive observers. Early work on stable adaptive observers for linear time-invariant systems can be found in [1, 2]. In [3], the linear adaptive observer was extended to a class of nonlinear time-varying systems, which can be transformed into an adaptive observer canonical form

$$\dot{x}(t) = Rx(t) + \Omega(\omega(t))\theta(t) + g(t), \; y(t) = x_1(t) \qquad (1)$$

where $\omega(t)$ is a vector of known functions of $u(t)$ and $y(t)$, and $\Omega(\omega(t))$ is a known $n \times p$ matrix, θ is a vector of unknown parameters, $g(t)$ is a vector of known functions, and R is a known constant matrix. Note that system (1) is linear in the unknown parameters and nonlinearities are restricted to be functions of the known input and output signals. An adaptive observer, driven by a $p(n-1)$ dimensional auxiliary filter, was developed for (1); stable convergence of the estimates was shown under certain persistency of excitation conditions.

Necessary and sufficient conditions for transforming a general nonlinear system into a canonical form that is nonlinear purely in the output variables can be found in [4]. Based on the early work of [3, 5], considerable work on adaptive nonlinear observers was reported by Marino et. al. in a series of papers; see [6] and the references there-in; Marino et. al. studied adaptive observers for a class of nonlinear systems that can be transformed via a global state space diffeomorphism into

$$\begin{aligned} \dot{x}(t) &= A_c x(t) + \psi_0(y(t), u(t)) + b\psi^T(y(t), u(t))\theta \\ y(t) &= C_c x(t) \end{aligned} \qquad (2)$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}$, $u(t) \in \mathbb{R}^m$, $\psi(y(t), u(t)) \in \mathbb{R}^p$ is a known smooth function of the output, $y(t)$, and the input vector, $u(t)$, $\theta \in \mathbb{R}^p$ is an unknown parameter vector, and $A_c = \begin{bmatrix} 0 & I_{n-1} \\ 0 & 0 \end{bmatrix}$, $C_c = [1, 0, \cdots, 0]$. Notice that the system is linear in the unknown parameters and the nonlinearities are functions of the known output and input variables only.

In [7], design of a nonlinear observer for nonlinear systems using Lyapunov's auxiliary theorem was proposed; the nonlinear observer design was analogous to the linear Luenberger observer theory. A dual-observer structure to estimate the unmeasurable state of the Lugre dynamic friction model was proposed in [8]; an adaptive controller and observer was designed to simultaneously estimate the unknown friction parameters and the unmeasurable friction state.

Motivated by some practical real-world examples and also to construct stable adaptive observers for a broader class of systems, we consider the following class of systems that contain the product of the unmeasurable state variables as well as unknown parameters:

$$\begin{aligned} \dot{x}(t) &= Mx(t) + hu(t) + h[d(x(t)) + f_\beta^\top(x(t))\beta \\ &\quad + f_\xi^\top(x(t))\xi(t) + \beta^\top G_\xi(x(t))\xi(t)], \\ \dot{\xi}(t) &= a_\xi(x(t)) + B_\xi(x(t))\xi(t) \end{aligned} \qquad (3)$$

where $x(t) \in \mathbb{R}^n$ is the measurable state, $\xi(t) \in \mathbb{R}^m$ is the unmeasurable state, $u(t) \in \mathbb{R}$ is the control input, $\beta \in \mathbb{R}^p$ is an unknown constant parameter vector, $M \in \mathbb{R}^{n \times n}$ is a known constant matrix, $h \in \mathbb{R}^n$ is a known constant vector, and $d(x(t)) \in \mathbb{R}^n$, $f_\beta(x(t)) \in \mathbb{R}^p$, $f_\xi(x(t)) \in \mathbb{R}^m$, $G_\xi(x(t)) \in \mathbb{R}^{p \times m}$, $a_\xi(x(t)) \in \mathbb{R}^m$, $B_\xi(x(t)) \in \mathbb{R}^{m \times m}$ are known functions of $x(t)$; $f^\top$ denotes the transpose of f. Notice that the last term in the x-dynamics in (3) is a product of the unmeasurable state and the unknown parameter vector.

The adaptive controller and observer design process first involves casting the system described by (3) into a certain suitable form, which is given in Section 2 below; this modified form of the dynamics consists of a new unmeasurable state

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that is formed by stacking of a number of unmeasurable state vectors $\xi(t)$; further the modified form of the dynamics also contains a new parameter vector that is formed by stacking of the elements of the original parameter vector β in a certain way. In this modified form the new parameter vector and the new unmeasurable state are of the same dimension, which is much larger than both m and p. The process of expressing (3) into the new form is constructive and is always possible. Based on this new form, Lyapunov's method is used to design an adaptive controller and observer. The proposed adaptive controller and observer is shown to result in asymptotic regulation of the measurable state, asymptotic convergence of the state estimation errors to zero or their boundedness based on certain conditions, and boundedness of the estimated parameters. Simulation results on two distinct examples are shown and discussed for the proposed scheme.

The rest of the paper is organized as follows. Section 2 gives the formulation of the dynamics of (3) into the modified form that is suitable for adaptive controller and observer design; it also gives the control objective. Section 3 shows the design of the adaptive controller and nonlinear observer using Lyapunov's method for a simple example. The design for the general case is presented in Section 4. Simulation results using the proposed scheme for two examples are given in Section 5. Conclusions and future research are given in Section 6.

2 Formulation

In this section, the system described by (3) is expressed in a modified form that is suitable for the proposed adaptive controller and observer design. First, we assume that the system dynamics described by (3) satisfies the following:

1) The pair (M, h) is controllable.
2) The ξ-dynamics is input-to-state stable in x [9]. The matrix $B_\xi(x) \in \mathbb{R}^{m\times m}$ is diagonal and $B_\xi(x) \le -\epsilon I$ for all $x \in \mathbb{R}^n$, where ϵ is a non-negative constant and I denotes the $m \times m$ identity matrix.
3) The sign of each parameter, $\beta_i, i = 1 : p$, in the parameter vector β is known, and β_i is bounded.
4) The functions $d(x)$, $f_\beta(x)$, $f_\xi(x)$, $G_\xi(x)$, $a_\xi(x)$ and $B_\xi(x)$ are bounded if x is bounded.

Assumption 1) guarantees the existence of a control gain vector $c \in \mathbb{R}^n$ that can stabilize the linear part of the x-dynamics; that is, there exists a symmetric positive definite solution, P, to the Lyapunov equation $(M - hc^\top)^\top P + P(M - hc^\top) = -Q$, where Q is a symmetric positive definite matrix. Assumption 2) ensures that ξ has stable dynamics, and there is no coupling between unmeasurable states; in fact we just require the matrix $B_\xi(x)$ to be diagonalizable using a linear transformation; this includes the case where the matrix $B_\xi(x)$ is constant. Assumption 3) is the only required knowledge of the unknown parameters and is reasonable for many practical plants; it allows us to use a parameter dependent Lyapunov function candidate during the design process.

Nonlinear system (3) can be cast in the following form:

$$\begin{aligned}\dot{x} &= Mx + hu + h[d(x) + f^\top(x)\theta + z^\top G(x)\theta],\\ \dot{z} &= a(x) + B(x)z\end{aligned} \tag{4}$$

where $z^\top = [\underbrace{\xi^\top, \cdots, \xi^\top, \cdots, \xi^\top}_{(p+1)-times}]$, $\theta^\top = [\underbrace{\beta_1, \cdots, \beta_1}_{m-times}, \cdots, \underbrace{\beta_p, \cdots, \beta_p}_{m-times}, \underbrace{1, \cdots, 1}_{m-times}]$, $f^\top(x) = [f_{\beta_1}(x), \underbrace{0, \cdots, 0}_{(m-1)-times}, \cdots, f_{\beta_p}(x), \underbrace{0, \cdots, 0}_{(2m-1)-times}]$, $a^\top(x) = [\underbrace{a_\xi^\top(x), \cdots, a_\xi^\top(x)}_{(p+1)-times}]$, $B(x) = \mathrm{diag}(\underbrace{B_\xi(x), \cdots, B_\xi(x)}_{(p+1)-times})$, $G(x) = \mathrm{diag}(g_{\xi 11}(x), \cdots, g_{\xi 1m}(x), \cdots, g_{\xi p1}(x), \cdots, g_{\xi pm}(x), f_{\xi 1}(x), \cdots, f_{\xi m}(x))$ where $g_{\xi ij}(x)$, $i = 1 : p$, $j = 1 : m$, is the ij-th element of $G_\xi(x)$, and $f_{\xi i}(x)$, $i = 1 : m$, is the i-th element of $f_\xi(x)$, $f_{\beta i}$, $i = 1 : p$, is the i-th element of $f_\beta(x)$.

The main difference between the models described by (3) and (4) is that the unknown parameter vector θ and the unmeasurable state vector z are of larger dimension than that of β and ξ; $z \in \mathbb{R}^{m(p+1)}$ is a vector cascaded by $(p+1)$ ξ's; $\theta \in \mathbb{R}^{m(p+1)}$ is a vector cascaded by m times β_i's, $i = 1 : p$, and the m-vector with each entry equal to 1; $a(x) \in \mathbb{R}^{m(p+1)}$ is cascaded by $(p+1)$ $a_\xi(s)$'s; $G(x) \in \mathbb{R}^{m(p+1)\times m(p+1)}$ is a diagonal matrix whose diagonal entries are the entries of $G_\xi(x)$ and $f_\xi(x)$; $f(x) \in \mathbb{R}^{m(p+1)}$ is a vector whose $[(j-1)m+1]$-th element is $f_{\beta j}(x)$, $j = 1 : p$ and the other elements are zero.

The motivation for casting the nonlinear system (3) in the form given by (4) with a new parameter vector θ and a new state vector z is to account for the non-zero off-diagonal entries in $G_\xi(x)$ and non-zero entries in $f_\xi(x)$; as a result of this, the proposed stable adaptive controller and observer design is feasible. A non-zero off-diagonal entry in $G_\xi(x)$ means that two unknown parameters are coupled with the same unmeasurable state variable (or two unmeasurable state variables are coupled with the same parameter). In such a case, two different observers are needed to estimate the same unmeasured state variable (or two parameter estimation algorithms are needed for one unknown parameter). The situation is the same if an unmeasured state is coupled with an unknown parameter and also appears linearly in the function of x.

Assuming that none of the elements of matrix $G_\xi(x)$ and vector $f_\xi(x)$ are zero, the maximum number of state variables that should be estimated is $m(p+1)$ and the maximum number of parameters that should be estimated is mp. Notice that the maximum size of z (or θ) is $m(p+1)$. If some entries in $G_\xi(x)$ and/or $f_\xi(x)$ are zero, it is possible to reduce the dimension of the vector z. Correspondingly, the dimensions of $G(x)$, θ, $f(x)$, $a(x)$ and $B(x)$ are also reduced. Assume that the size of z (or θ) after reduction is q. The following example illustrates the reduction procedure. Consider a system with $m = 2$ and $p = 3$. The modified system has $z^\top = [\xi_1, \xi_2, \xi_1, \xi_2, \xi_1, \xi_2, \xi_1, \xi_2]$, $\theta^\top = [\beta_1, \beta_1, \beta_2, \beta_2, \beta_3, \beta_3, 1, 1]$, $G(x) = \mathrm{diag}(g_{\xi 11}(x), g_{\xi 12}(x), g_{\xi 21}(x), g_{\xi 22}(x), g_{\xi 31}(x), g_{\xi 32}(x), f_{\xi 1}(x), f_{\xi 2}(x))$, $a^\top(x) = [a_{\xi 1}(x), a_{\xi 2}(x), a_{\xi 1}(x), a_{\xi 2}(x), a_{\xi 1}(x), a_{\xi 2}(x), a_{\xi 1}(x), a_{\xi 12}(x)]$, $B(x) = \mathrm{diag}(B_\xi(x),$

$B_\xi(x), B_\xi(x), B_\xi(x)), f^\top(x) = [f_{\beta 1}(x), 0, f_{\beta 2}(x), 0, f_{\beta 3}(x), 0, 0, 0]$. If $f_\xi^\top(x) = [0,\ 0]$, discard the last two rows of z, θ, $a(x)$ and $f(x)$, and the last two rows and columns of $G(x)$ and $B(x)$ resulting in $q = 6$, which is less than the maximum size, eight. If $g_{\xi 12}(x) = 0$, then the second row of z, θ, $a(x)$ and $f(x)$, and the second row and the second column of $G(x)$ and $B(x)$ can be discarded, which gives $q = 7$.

With the assumptions on the original system described by (3), it is easy to see that the following three assumptions, which correspond to assumption 2), 3), and 4), respectively, are true for the system in the modified form (4):

2′) The z-dynamics is input-to-state stable in x. The matrix $B(x)$ is diagonal and $B(x) \le -\epsilon I$ for all $x \in \mathbb{R}^n$ and a non-negative ϵ.

3′) The sign of each parameter, $\theta_i, i = 1 : q$, in the parameter vector θ is known, and θ_i is bounded.

4′) The functions $d(x)$, $f(x)$, $G(x)$, $a(x)$ and $B(x)$ are bounded if x is bounded.

3 Adaptive controller and observer design: A simple example

In this section, a simple example is considered to show the design process using Lyapunov's method. The example is given by the following equations:

$$\begin{aligned} \dot{x} &= u + f(x)\theta + g(x)z\theta, \\ \dot{z} &= b(x)z \end{aligned} \tag{5}$$

where $x, u, \theta, z, f(x), g(x), b(x) \in \mathbb{R}$. The sign of θ is known and $b(x) \le -\epsilon$, $\epsilon > 0$, for all $x \in \mathbb{R}$. The goal is to design a control algorithm such that x converges asymptotically to zero; this involves design of the control input u, observer design to estimate z, and design of an adaptation scheme for θ.

Choose the following control input:

$$u = -cx - f(x)\widehat{\theta} - g(x)\widehat{z}\widehat{\theta} \tag{6}$$

where $c > 0$. Substituting the control input (6) into the x-dynamics of (5) results in the equation:

$$\dot{x} = -cx - f(x)\widetilde{\theta} - g(x)\widehat{z}\widetilde{\theta} - g(x)\widetilde{z}\theta \tag{7}$$

where $\widehat{(*)}$ is the estimate of $(*)$, and $\widetilde{(*)} = \widehat{(*)} - (*)$ is the estimation error of $(*)$. To design an observer for z and a parameter adaptation algorithm for θ, the following Lyapunov function candidate is chosen:

$$V = \frac{1}{2}x^2 + V_{\widetilde{\theta}} + V_{\widetilde{z}} \tag{8}$$

where $V_{\widetilde{\theta}}$ is a radially unbounded positive function of $\widetilde{\theta}$ and $V_{\widetilde{z}}$ is a positive function of $\widetilde{z}$ and θ, and is radially unbounded with respect to $\widetilde{z}$. The time derivative of V along (6) is

$$\dot{V} = -cx^2 - f(x)x\widetilde{\theta} - g(x)x\widehat{z}\widetilde{\theta} - g(x)x\widetilde{z}\theta + \dot{V}_{\widetilde{\theta}} + \dot{V}_{\widetilde{z}}. \tag{9}$$

A sufficient condition for $\dot{V} \le -cx^2$ is

$$\dot{V}_{\widetilde{\theta}} + \dot{V}_{\widetilde{z}} \le f(x)x\widetilde{\theta} + g(x)x\widehat{z}\widetilde{\theta} + g(x)x\widetilde{z}\theta. \tag{10}$$

One may choose $\dot{V}_{\widetilde{\theta}}$ and $\dot{V}_{\widetilde{z}}$ to satisfy inequality (10) as

$$\dot{V}_{\widetilde{\theta}} = f(x)x\widetilde{\theta} + g(x)x\widehat{z}\widetilde{\theta}, \tag{11}$$

$$\dot{V}_{\widetilde{z}} \le g(x)x\widetilde{z}\theta. \tag{12}$$

The following choice of $V_{\widetilde{\theta}}$ and $\dot{\widetilde{\theta}}$ can satisfy (11):

$$V_{\widetilde{\theta}} = \frac{1}{2\gamma_1}\widetilde{\theta}^2, \quad \dot{\widetilde{\theta}} = \gamma_1(f(x) + g(x)\widehat{z})x \tag{13}$$

where $\gamma_1 > 0$. Equation (12) can be rewritten as

$$\frac{\partial V_{\widetilde{z}}}{\partial \widetilde{z}}\dot{\widetilde{z}} \le g(x)x\widetilde{z}\theta. \tag{14}$$

Substituting the z-dynamics of (5) into (14) yields

$$\frac{\partial V_{\widetilde{z}}}{\partial \widetilde{z}}(\dot{\widehat{z}} - b(x)z) \le g(x)x\widetilde{z}\theta. \tag{15}$$

Now consider the following choice for $V_{\widetilde{z}}$:

$$V_{\widetilde{z}} = \frac{1}{2\gamma_2}|\theta|\widetilde{z}^2 \tag{16}$$

where $\gamma_2 > 0$, and $|\theta| > 0$ is assumed. Equation (15) becomes

$$\widetilde{z}[\dot{\widehat{z}} - b(x)z - \gamma_2\,\mathrm{sgn}(\theta)g(x)x] \le 0. \tag{17}$$

Therefore, $\dot{V}_{\widetilde{z}} = b(x)\widetilde{z}^2 \le 0$ if

$$\dot{\widehat{z}} = b(x)\widehat{z} + \gamma_2\,\mathrm{sgn}(\theta)g(x)x \tag{18}$$

Thus, $\dot{V} = -cx^2 + \frac{1}{\gamma_2}b(x)|\theta|\widetilde{z}^2 \le -cx^2 - \frac{1}{\gamma_2}\epsilon|\theta|\widetilde{z}^2$. Hence, $x(t)$ and $\widetilde{z}(t)$ converge asymptotically to zero, and the parameter estimate $\widehat{\theta}(t)$ is bounded.

4 Adaptive controller and observer design: The general case

The design process outlined in the previous section for the simple example can be extended to the general case given by (4). The following theorem gives the result.

Theorem 1 *Consider the system (4). The following control law (19), parameter estimation algorithm (20), and observer (21) guarantee that $x(t)$ converges asymptotically to zero.*

$$u = -c^\top x - d(x) - f^\top(x)\widehat{\theta} - \widehat{z}^\top G(x)\widehat{\theta}, \tag{19}$$

$$\dot{\widehat{\theta}} = 2\Gamma[G(x)\widehat{z} + f(x)]h^\top Px, \tag{20}$$

$$\dot{\widehat{z}} = a(x) + B(x)\widehat{z} + P_z^{-1}G(x)sgn(\theta)h^\top Px \tag{21}$$

where $\Gamma = \Gamma^\top \in \mathbb{R}^{q\times q} > 0$, $c \in \mathbb{R}^n$, $P_z \in \mathbb{R}^{q\times q} > 0$ and is diagonal, $sgn(\theta) = [sgn(\theta_1), \cdots, sgn(\theta_q)]^\top$, and c is chosen such that P is the symmetric positive definite solution of the Lyapunov equation, $(M - hc^\top)^\top P + P^\top(M - hc^\top) = -Q$, for any given positive definite matrix Q. Further, $\widehat{\theta}$ and $\widehat{z}$ are bounded, and if $\epsilon \ne 0$, then $\lim_{t\to\infty}\widehat{z}(t) = z(t)$.

Proof. Using (19), (20) and (21), we have

$$\dot{x} = (M\text{-}hc^\top)x - h(f^\top(x)+\widetilde{z}^\top G(x))\widetilde{\theta} - h\theta^\top G(x)\widetilde{z}, \quad (22)$$

$$\dot{\widetilde{z}} = B(x)\widetilde{z} + P_z^{-1}G(x)sgn(\theta)h^\top Px. \quad (23)$$

Consider the following Lyapunov function candidate:

$$V(x,\widetilde{\theta},\widetilde{z},\theta) = x^\top Px + \frac{1}{2}\widetilde{\theta}^\top\Gamma^{-1}\widetilde{\theta} + \widetilde{z}^\top\Lambda_{|\theta|}P_z\widetilde{z} \quad (24)$$

where $\Lambda_{|\theta|}$ is a diagonal matrix whose i-th diagonal element is the absolute value of the i-th element of the parameter vector θ, i.e., $\Lambda_{|\theta|} = \mathrm{diag}(|\theta_1|,\cdots,|\theta_q|)$. The time derivative of V is

$$\begin{aligned}\dot{V} &= \dot{x}^\top Px + x^\top P\dot{x} + \widetilde{\theta}^\top\Gamma^{-1}\dot{\widetilde{\theta}} + 2\widetilde{z}^\top\Lambda_{|\theta|}P_z\dot{\widetilde{z}} \\ &= x^\top[(M-hc^\top)^\top P + P(M-hc^\top)]x + 2\widetilde{z}^\top\Lambda_{|\theta|}P_z \\ &\quad B(x)\widetilde{z} + 2\widetilde{z}^\top(\Lambda_{|\theta|}G(x)\,\mathrm{sgn}(\theta) - G(x)\theta)h^\top Px \\ &\leq -x^\top Qx - 2\epsilon\widetilde{z}^\top\Lambda_{|\theta|}P_z\widetilde{z} \end{aligned} \quad (25)$$

where $\dot{\widehat{\theta}} = \dot{\widetilde{\theta}}$, $(M-hc^\top)^\top P + P(M-hc^\top) = -Q$, assumption (2′), equations (20), (22), and (23), and the property $\Lambda_{|\theta|}G(x)\,\mathrm{sgn}(\theta) = G(x)\theta$ are applied. Hence, (24) is a Lyapunov function for the closed-loop system, which guarantees that $x(t)$, $\widetilde{\theta}(t)$ and $\widetilde{z}(t)$ are bounded; $\widehat{\theta}(t)$ is bounded because $\widehat{\theta}(t) = \widetilde{\theta}(t) + \theta$ and θ is bounded; $z(t)$ is bounded by assumption 2) and 4), which in turn guarantees that $\widehat{z}(t)$ $(= z(t) + \widetilde{z}(t))$ is bounded; the control input is bounded as it is a function of all bounded variables. From equations (22) and (23), both $\dot{\widetilde{z}}(t)$ and $\dot{x}(t)$ are bounded. Therefore, $\widetilde{z}(t) \in \mathcal{L}_\infty$, $\dot{\widetilde{z}}(t) \in \mathcal{L}_\infty$, $x(t) \in \mathcal{L}_\infty \cap \mathcal{L}_2$ and $\dot{x}(t) \in \mathcal{L}_\infty$. By invoking Barbalat's Lemma, we obtain $\lim_{t\to\infty} x(t) = 0$. If $\epsilon \neq 0$, then $\widetilde{z}(t) \in \mathcal{L}_2$; therefore, $\lim_{t\to\infty}\widetilde{z}(t) = 0$. □

Remark 1 *Theorem* 1 *addresses the regulation problem for the class of nonlinear systems described by (3). This design process can be applied to the tracking problem as well, which is shown in example 2 of the next section.*

Remark 2 *From equations (19), (20), and (21) notice that if the i-th diagonal element of the matrix $G(x)$ is zero, then there is no need to estimate the state variable $z_i(t)$. Again, example 2 of the next section illustrates this aspect.*

Remark 3 *Notice that the original system described by* (3) *has m state variables that are to be estimated and p parameters that are to be estimated; but in the proposed design, if none of the elements of the vector $f_\xi(x)$ and the matrix $G_\xi(x)$ is zero, we need to have $m(p+1)$ filters for the estimation of the unmeasurable states and mp filters for the estimation of the unknown parameters.*

5 Simulation

In this section, two examples are presented to verify the adaptive controller and observer design. In the first example, the system has two measurable states, two unmeasurable states and two unknown parameters. The objective is regulation of states. In the second example, we consider the tracking problem for a mechanical system with dynamic friction.

Example 1. Consider the system

$$\begin{aligned}\dot{x} &= Mx + hu + h(f_\beta^\top(x)\beta + f_\xi^\top(x)\xi + \beta^\top G_\xi(x)\xi), \\ \dot{\xi} &= a_\xi(x) + B_\xi(x)\xi \end{aligned} \quad (26)$$

where $M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $h = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $f_\beta(x) = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$, $a_\xi(x) = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix}$, $G_\xi(x) = \begin{bmatrix} \frac{x_1}{2-e^{-|x_2|}} & 5 \\ \frac{x_2}{2-e^{-|x_1|}} & 0 \end{bmatrix}$, $f_\xi(x) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$, $B_\xi(x) = \begin{bmatrix} -1.5-\cos(x_1) & 0 \\ 0 & -1.5-\sin(x_2) \end{bmatrix}$. The system described by (26) can be represented in the form of (4) with $d(x) = 0$, $f^\top(x) = [f_{\beta 1}, 0, f_{\beta_2}, 0, 0]$, $G(x) = \mathrm{diag}(g_{\xi 11}, g_{\xi 12}, g_{\xi 21}, f_{\xi 1}, f_{\xi 2})$, $a^\top(x) = [a_{\xi 1}, a_{\xi 2}, a_{\xi 1}, a_{\xi 1}, a_{\xi 2}]$, $B(x) = \mathrm{diag}(b_{\xi 11}, b_{\xi 22}, b_{\xi 11}, b_{\xi 11}, b_{\xi 22})$, $z^\top = [z_1,\cdots,z_5]$, $\theta^\top = [\theta_1,\cdots,\theta_5]$, and $\theta_4 = \theta_5 = 1$ which are not estimated. The following values are chosen: $c^\top = [25, 10]$, $P_z = 10I$, $\Gamma = 0.1I$ where I is the 5×5 identity matrix. The true parameter vector is $\beta^\top = [2, 4]$. The following initial values are chosen: $x^\top(0) = [1, 1]$, $\widehat{\beta}^\top(0) = [0, 0, 0, 1, 1]$, $z^\top(0) = [-1, -1, -1, -1, -1]$ and $\widehat{z}^\top(0) = [0, 0, 0, 0, 0]$. Simulation results are shown in Figures 1 through 3; it can be observed that $x(t)$ asymptotically converges to zero; the estimated state $\widehat{z}(t)$ converges to $z(t)$; and the estimated parameter $\widehat{\theta}(t)$ is bounded.

Example 2. Consider a single-link mechanical system described by

$$J\ddot{s} = u - f_f \quad (27)$$

where J is the inertia of the link, s is the angular position of the link, $\dot{s}$ is the angular velocity of the link, u is the control input, and f_f is the friction torque described by the following LuGre dynamic friction model [10]:

$$f_f = \beta_1\xi + \beta_2\dot{\xi} + \beta_3\dot{s}, \quad (28)$$

$$\dot{\xi} = \dot{s} - \frac{\sigma|\dot{s}|}{g(\dot{s})}\xi,\ g(\dot{s}) = F_c + (F_s - F_c)e^{-(\dot{s}/\omega_s)^2} \quad (29)$$

where σ, β_1, β_2, β_3, F_s, F_c and ω_s are positive friction coefficients; σ, F_s, F_c and ω_s are generally identified by experiments off-line and are assumed to be known for this simulation. J is known and is assumed to be equal to one. The objective is to control the link such that the position and velocity of the link track a predefined trajectory $s_d(t)$ and $\dot{s}_d(t)$, respectively. It is assumed that that $s_d(t)$ and $\dot{s}_d(t)$ are bounded, and the angular position and the angular velocity are measurable.

Combining (27), (28) and (29) and representing in matrix form yields

$$\dot{\zeta} = M\zeta + hu + h(f_\beta^\top\beta + \beta^\top G_\xi(\zeta)\xi), \quad (30)$$

$$\dot{\xi} = a_\xi(\zeta) + B_\xi(\zeta)\xi \quad (31)$$

where $\zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} s \\ \dot{s} \end{bmatrix}$, $M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $f_\beta = \begin{bmatrix} 0 \\ -\zeta_2 \\ -\zeta_2 \end{bmatrix}$,

$h = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $G_\xi(\zeta) = \begin{bmatrix} -1 \\ \frac{\sigma|\zeta_2|}{g(\zeta_2)} \\ 0 \end{bmatrix}$, $\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$, $a_\xi(\zeta) = \zeta_2$, $B_\xi(\zeta) = -\frac{\sigma|\zeta_2|}{g(\zeta_2)}$.

Defining the trajectory vector $x_d = [s_d, \ \dot{s}_d]$ and representing (30) and (31) in terms of the tracking error $x := [x_1, \ x_2]^\top = \zeta - x_d$ results in the following error dynamics and ξ-dynamics:

$$\dot{x} = Mx + hu + h(f_\beta^\top \beta + \beta^\top G_\xi(x + x_d)\xi), \tag{32}$$

$$\dot{\xi} = a_\xi(x + x_d) + B_\xi(x + x_d)\xi. \tag{33}$$

The above two equations can be re-written in the following form suitable for the adaptive controller and observer design:

$$\dot{x} = Mx + hu + h(f^\top(x)\theta + z^\top G(x)\theta), \tag{34}$$

$$\dot{z} = a(x) + B(x)z \tag{35}$$

where $\theta^\top = [\beta_1, \beta_2, \beta_3]$, $z^\top = [z_1, z_2, z_3]$, $f(x)^\top = [0, -x_2 - \dot{s}_d, -x_2 - \dot{s}_d]$, $G(x) = \mathrm{diag}(-1, \frac{\sigma|x_2+\dot{s}_d|}{g(x_2+\dot{s}_d)}, 0)$, $B(x) = \mathrm{diag}\left(-\frac{\sigma|x_2+\dot{s}_d|}{g(x_2+\dot{s}_d)}, -\frac{\sigma|x_2+\dot{s}_d|}{g(x_2+\dot{s}_d)}, -\frac{\sigma|x_2+\dot{s}_d|}{g(x_2+\dot{s}_d)}\right)$, and $a(x)^\top = [x_2 + \dot{s}_d, x_2 + \dot{s}_d, x_2 + \dot{s}_d]$. Notice that x $(= \zeta - x_d)$ is available because ζ is measurable and x_d is known; and θ and z are estimated; z_3 need not be estimated because $g_{33}(x) = 0$.

The following values are chosen for the simulations: $c^\top = [2500, 100]$, $\Gamma = \mathrm{diag}(500, 10, 10)$. $P_z = 0.01I$. The parameters in the simulation are: $\sigma = 50$, $F_s = 2$, $F_c = 0.5$, $\omega_s = 0.01$, $\beta_1 = 1000$, $\beta_2 = 100$, $\beta_3 = 0.1$. The following initial values are chosen: $x^\top(0) = [0, 0]$, $\widehat{\beta}^\top(0) = [0, 0, 0]$, $z^\top(0) = [0, 0, 0]$ and $\widehat{z}^\top(0) = [0, 0, 0]$. The desired trajectory is chosen to be $x_d^\top(t) = [\sin(\pi t), \pi\cos(\pi t)]$, which is shown in Figure 4.

The simulation results are shown in Figure 5 through 7. Figure 5 shows that the position and velocity tracking errors asymptotically converge to zero. The unmeasurable states, $z_1(t)$ and $z_2(t)$, and their estimates, $\widehat{z}_1(t)$ and $\widehat{z}_2(t)$, are shown in Figure 6; $z_1(t)$ and $z_2(t)$ have the same trajectory because they have the same dynamics. Notice that $B(x)$ in this example is negative semi-definite because $B(x)$ is identically zero when the velocity of the system is zero. Although the convergence of the estimation error, $\widetilde{z}(t)$, to zero is not guaranteed theoretically, the state estimates $\widehat{z}_1(t)$ and $\widehat{z}_2(t)$ appear to converge to $z_1(t)$ and $z_2(t)$, respectively, in the simulations. The estimated parameters, $\widehat{\beta}_1(t)$, $\widehat{\beta}_2(t)$ and $\widehat{\beta}_3(t)$, shown in Figure 7, are bounded.

6 Conclusions

A stable adaptive controller and a nonlinear observer were designed for a class of nonlinear systems using a parameter dependent Lyapunov function; the proposed design is for a more general class of nonlinear systems than the ones existing in the literature in the sense that the systems considered in this paper allow for the product of the unmeasured state and the unknown parameter. The proposed design was verified via two simulation examples, the results of which were shown and discussed. Future research should focus on relaxing the assumptions on the nonlinear system; of particular importance is the one requiring that there be no coupled terms in the unmeasurable state dynamics. Also, inclusion of unknown parameters in the unmeasurable state dynamics presents a challenging problem.

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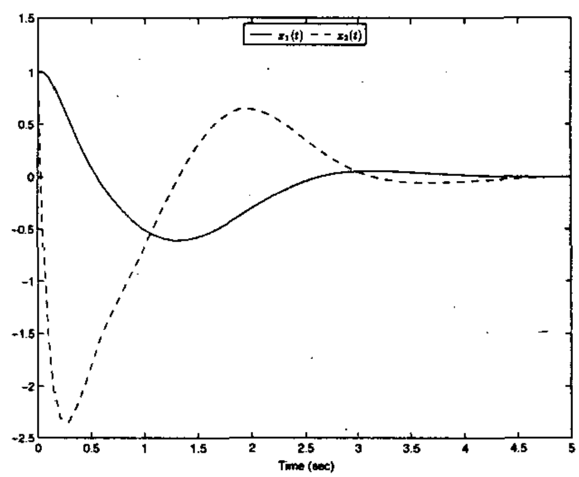

Figure 1: Example 1. Trajectory of $x(t)$.

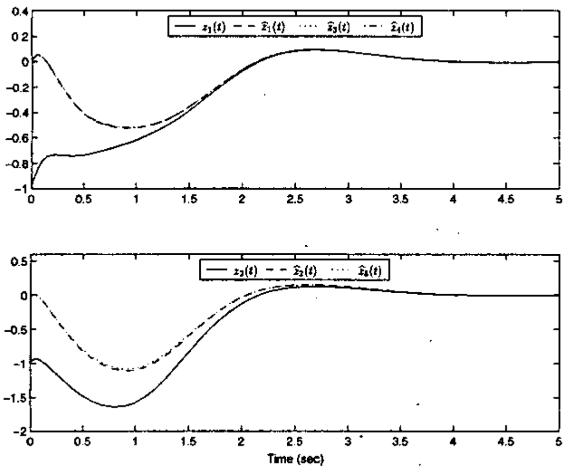

Figure 2: Example 1. Actual state $z(t)$ and its estimate $\widehat{z}(t)$.

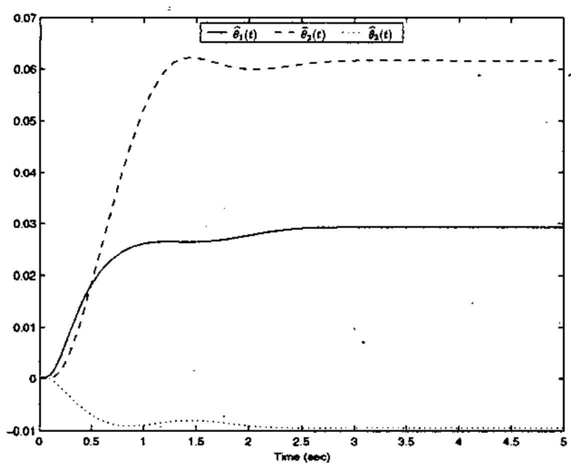

Figure 3: Example 1. Estimated parameters: $\widehat{\theta}_1(t)$, $\widehat{\theta}_2(t)$, $\widehat{\theta}_3(t)$ and $\widehat{\theta}_4(t)$.

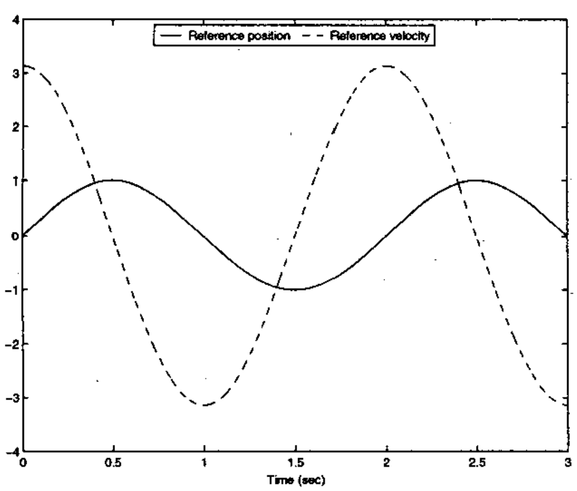

Figure 4: Example 2. Desired position and velocity.

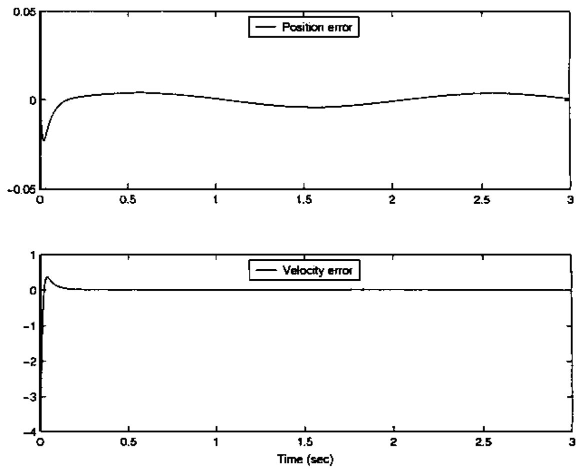

Figure 5: Example 2. Position and velocity tracking errors.

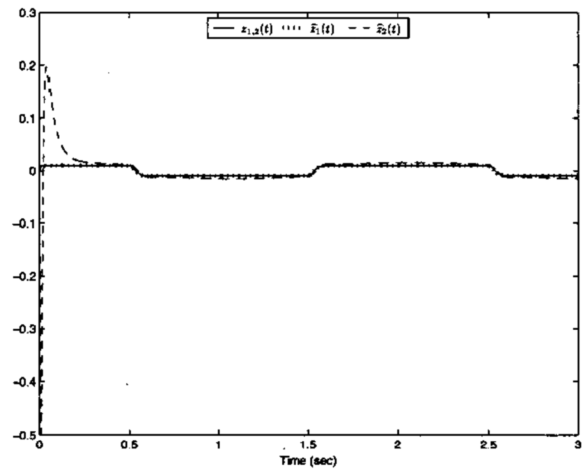

Figure 6: Example 2. Actual state $z(t)$ and its estimate $\widehat{z}(t)$.

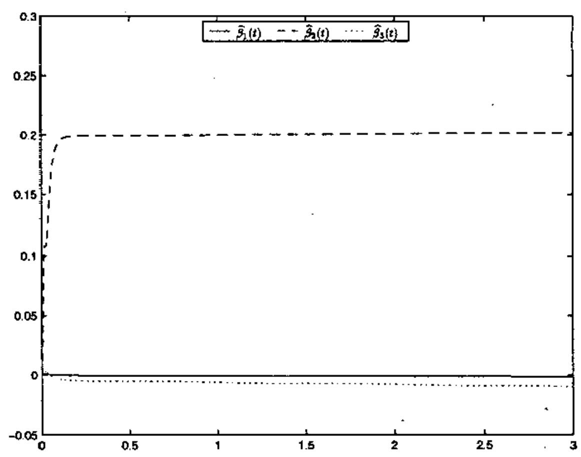

Figure 7: Example 2. Estimated parameters: $\widehat{\beta}_1(t), \widehat{\beta}_2(t)$ and $\widehat{\beta}_3(t)$.