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Control of Contact Problem in Constrained Euler–Lagrange Systems

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***Abstract*—Stabilization of an Euler–Lagrange system onto a constraint surface when the system makes contact with a nonzero impact velocity is an important problem in systems interacting with external environments. Potential applications include robotic surface following and surface finishing operations in manufacturing industry. In this paper, the constrained dynamic equations are modeled as a set of nonsmooth differential equations depending on whether the system lies on the constraint surface or the system repeatedly makes and loses contact with the constraint surface. The focus is on the initial condition problem, i.e., the system hits the constraint with a nonzero impact velocity. A new discontinuous control scheme is proposed that ensures stable regulation of the system onto the constraint surface.**

***Index Terms*—Discontinuous control, Euler–Lagrange systems, impact, nonsmooth Lyapunov analysis, unilateral constraint.**

I. INTRODUCTION

In this note, we consider systems described by Euler–Lagrange equations with a single unilateral constraint. Let the kinetic and potential energy of the Euler–Lagrange system be given by $\mathcal{K}(x, \dot{x}) = (1/2)\dot{x}^T M(x)\dot{x}$ and $\mathcal{P}(x)$, respectively, where $x \in \mathbb{R}^n$ is the generalized position, $\dot{x} \in \Re^n$ is the generalized velocity, and $M(x) \in \Re^{n \times n}$ is the symmetric positive–definite inertia matrix. Let $u \in \Re^n$ be the input forces, and $f_\mu(x, \dot{x}) \in \Re^n$ denote the nonconservative forces resulting from the discontinuous elements like friction, dead zones, etc. Let the single constraint be given by $\phi(x) - d \geq 0$, where d is a constant. The constraint takes the strict inequality form when the system is not in contact with the surface. When in contact with the surface, the constraint forces on the system are given by $f_\phi(x, \lambda) = g_\phi^T(x)\lambda$, where $g_\phi(x) := \nabla\phi(x)$ and λ is the Lagrange multiplier associated with the constraint. We assume that the constraint surface is rigid and frictionless. The Lagrangian for the system is $\mathcal{L}(x, \dot{x}) = \mathcal{K}(x, \dot{x}) - \mathcal{P}(x)$. The motion of the constrained system is described by the equations:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}(x, \dot{x})}{\partial \dot{x}}\right) - \left(\frac{\partial \mathcal{L}(x, \dot{x})}{\partial x}\right) = u + f_\mu(x, \dot{x}) + f_\phi(x, \lambda). \quad (1)$$

Assuming that the constrained Euler–Lagrange system is as described above, the problem that is investigated in this paper can be stated as follows: given a nonzero initial impact velocity of the system normal to the surface, i.e., $\phi(x) = 0$ and $g_\phi(x)\dot{x} \neq 0$, the control objective is to regulate the system onto the surface such that $\phi(x) = 0$ and $g_\phi(x)\dot{x} = 0$.

Research in characterizing the constrained dynamics of such systems has been active in the mechanics literature; see [1]–[3] and the references therein. Considerable research has been done in motion and force control of robots in constrained motion; see [31] and the references therein. Much of this research has been based on the assumption that the system is already in contact with the external environment. In many industrial applications such as robotic surface finishing, the robot moves in free motion before making contact with the surface. If the robot hits the surface with a nonzero impact velocity then transition from free motion to constrained motion involves impulsive forces on the system and the velocity variable is discontinuous. Uncertainty in constraint surface location could cause impact of the system with the surface. The system may experience repeated bounces on the surface depending on the magnitude of the impact velocity. Stable regulation of the system onto the constraint surface is required before applying motion and force control for the system to travel on the surface.

In literature, a number of controllers have been designed to deal with contact transitions. In [4], the environment is treated as a mechanical impedance and impedance control is used during contact transition. A dimensionless representation of impact behavior was developed in [5] and an integral force feedback is used to improve transient impact response. Stability and control of task transition for robots has been considered in [6] for a compliant environment, wherein the transitions are assumed to take place smoothly. Impact minimization by using redundant degrees of freedom in the robots has been considered in [7]. Force regulation and contact transition control is considered in [8]. A discontinuous control approach to general task execution of manipulators interacting with a compliant environment has been considered in [9]. In all these control algorithms, the transition is assumed to be smooth and/or an explicit impact model is used. In this paper, we describe the motion of the constrained Euler–Lagrange system as a set of nonsmooth differential equations. A new discontinuous control algorithm that ensures stable regulation of the system onto the constrained surface in finite time is proposed. Since the closed-loop equations are nonsmooth, results from nonsmooth Lyapunov theory [10]–[13] are used to show stability.

The proposed control algorithm can be designed for any surface as the impact model is not used in the design of the control algorithm. Impact phenomena has been extensively studied in literature; see [1], [14]–[17]. Researchers favor different impact models depending on the contact condition and the applications under consideration. Typically there are three distinct definitions of the coefficient of restitution: i)

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kinematic coefficient (Newton's coefficient), defined as a ratio of velocities, ii) kinetic coefficient (Poisson coefficient), defined as a ratio of impulses, and iii) energetic coefficient, defined as a ratio of energies. [14] favors the energetic coefficient whereas [17] uses the kinematic coefficient, and [18] favors the kinetic coefficient. See [16] for an overview of the impact models.

The rest of this note is organized as follows: In Section II, nonsmooth differential equations describing the constrained dynamics are developed. Control algorithm and stability of the closed-loop system is given in Section III. Section IV concludes this note with a summary and future work.

II. CONSTRAINED DYNAMICS OF EULER–LAGRANGE SYSTEMS

It is known in the mechanics literature [1], [3], [19] and the robotics literature [24], that the generalized position, x, can be transformed such that the constraint can be represented as a link constraint in one coordinate variable in the transformed coordinates. This is usually done as follows. The generalized position, x, can be partitioned as $x^T = (x_1, x_2^T)$, where $x_1 \in \Re^1$ and $x_2 \in \Re^{n-1}$. By the implicit function theorem [20], there exists a function $\sigma(\cdot)$ such that $x_1 = \sigma(x_2)$. Knowing that x_1 can be expressed in terms of x_2, a change of coordinates can be made by constructing a transformation, $\psi(\cdot): (x_1, x_2) \rightarrow (q, r)$. This transformation is given by $q = x_1 - \sigma(x_2)$, $r = x_2$, where $q \in \Re^1$ and $r \in \Re^{n-1}$. Notice that the inverse of this transformation is $x_1 = q + \sigma(r)$ and $x_2 = r$. The Jacobian of the inverse transformation is an $n \times n$ matrix denoted by $T(r)$, and relates the velocities $(\dot{x}_1, \dot{x}_2)$ and $(\dot{q}, \dot{r})$, that is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & \frac{\partial \sigma(r)}{\partial r} \\ 0 & I_{n-1} \end{bmatrix} \begin{bmatrix} \dot{q} \\ \dot{r} \end{bmatrix} \tag{2}$$

where I_{n-1} denotes an identity matrix of size $n-1$. In the transformed coordinates, when $q = 0$ the system is on the constraint surface. Without loss of generality, we assume that $q \geq 0$ lies in the work space of the system. If this is not so, then one can define $q = \sigma(x_2) - x_1$ such that $q \geq 0$. Also, in the transformed coordinates the input forces, constrained forces, and the friction forces are $T^T(r)u$, $T^T(r)f_\phi$, and $T^T(r)f_\mu$, respectively. The kinetic energy in the transformed coordinates is

$$\mathcal{K}^*(q, \dot{q}, r, \dot{r}) = \frac{1}{2} [\dot{q} \quad \dot{r}^T] \underbrace{\begin{bmatrix} a(q,r) & b^T(q,r) \\ b(q,r) & A(q,r) \end{bmatrix}}_{T^T(r)M(q,r)T(r)} \begin{bmatrix} \dot{q} \\ \dot{r} \end{bmatrix}. \tag{3}$$

Notice that in the above expression, the inertia matrix in the transformed coordinates is partitioned appropriately to suit the size of the transformed coordinates (q, r). Let $\mathcal{P}^*(q, r)$ represent the potential energy in the transformed coordinates, then the transformed Lagrangian is

$$\mathcal{L}^*(q, r, \dot{q}, \dot{r}) = \frac{1}{2} \left(a(q,r)\dot{q}^2 + 2\dot{q}b^T(q,r)\dot{r} + \dot{r}^T A(q,r)\dot{r} \right) - \mathcal{P}^*(q,r). \tag{4}$$

The constraint in the transformed coordinates is transparent, i.e., when $q = 0$ and $\dot{q} \neq 0$, then there is an impact. The following sets distinguish whether the system lies on the surface or away from the surface:

$$S_{ct} := \{q, \dot{q} \in \Re^1; r, \dot{r} \in \Re^{n-1}: q = 0, \dot{q} \neq 0\} \tag{5}$$

$$S_{ca} := \{q, \dot{q} \in \Re^1; r, \dot{r} \in \Re^{n-1}: q = 0, \dot{q} = 0\} \tag{6}$$

$$S_u := \{q, \dot{q} \in \Re^1; r, \dot{r} \in \Re^{n-1}: q > 0\}. \tag{7}$$

The system in the transformed coordinates is given by the Lagrangian $\mathcal{L}^*(q, r, \dot{q}, \dot{r})$, input forces $(u_q, u_r) := T^T(r)u$, nonconservative forces $(f_{\mu q}, f_{\mu r}) := T^T(r)f_\mu$, and the constraint on the system is $q \geq 0$. The constraint force vector in the transformed coordinates is $(f_{\phi q}, 0) := T^T(r)f_\phi$. Notice that the last $n-1$ elements of the vector $T^T(r)f_\phi$ are all zero due to the fact that the direction of the velocity $\dot{q}$ is normal to the constraint surface. Denote

$$c(q, \dot{q}, r, \dot{r}) := \frac{\partial \mathcal{L}^*}{\partial q} + \dot{a}(q,r)\dot{q} - f_{\mu q}(q, r, \dot{q}, \dot{r}) + \dot{b}^T(q,r)\dot{r}$$

$$C(q, \dot{q}, r, \dot{r}) := \frac{\partial \mathcal{L}^*}{\partial r} + \dot{A}(q,r)\dot{r} - f_{\mu r}(q, r, \dot{q}, \dot{r}) + \dot{b}(q,r)\dot{q}.$$

Notice that $c(q, \dot{q}, r, \dot{r}) \in \Re^1$ and $C(q, \dot{q}, r, \dot{r}) \in \Re^{n-1}$. Using Lagrange's equation of motion and with the knowledge of the sub-division of the state space, the following equations describe the dynamics of the constrained system:

$$\begin{bmatrix} a(q,r) & b^T(q,r) \\ b(q,r) & A(q,r) \end{bmatrix} \begin{bmatrix} \ddot{q} \\ \ddot{r} \end{bmatrix} + \begin{bmatrix} c(q, \dot{q}, r, \dot{r}) \\ C(q, \dot{q}, r, \dot{r}) \end{bmatrix} = \begin{bmatrix} u_q \\ u_r \end{bmatrix} + \begin{bmatrix} f_{\phi q} \\ 0 \end{bmatrix}. \tag{8}$$

If $\underline{(q, r) \in S_{ct}}$, then the jump condition for (8) is given by

$$\dot{q}_+ = \mathcal{D}(q, \dot{q}_-) \tag{9}$$

where $\dot{q}_+$ and $\dot{q}_-$ represent the post and pre-impact velocities, respectively, and $\mathcal{D}(\cdot)$ represents an operator which maps pre-impact velocity into post-impact velocity. This operator can take several forms depending on the choice of the impact model for the constraint surface. Generally, impacts are treated as very large forces acting over a short duration of time. If we assume that impact occurs over an infinitesimally small period of time, then (i) all velocities remain finite and (ii) there is no change in position of the system. Integrating (8) for the duration of impact, the dynamics during impact becomes

$$\begin{bmatrix} a(q,r) & b^T(q,r) \\ b(q,r) & A(q,r) \end{bmatrix} \begin{bmatrix} \dot{q}_+ - \dot{q}_- \\ \dot{r}_+ - \dot{r}_- \end{bmatrix} = \begin{bmatrix} f_{qI} \\ 0 \end{bmatrix} \tag{10}$$

where $\dot{r}_+$ and $\dot{r}_-$ represent post and preimpact velocities, respectively, and f_{qI} is the force impulse due to impact. Both the post impact velocity and the force impulse can be computed knowing the impact map. The impact map can be chosen based on an impact model. For the Newton's impact model, $\mathcal{D}(\cdot) = -\epsilon \dot{q}_-$, where ϵ is the coefficient of restitution. In this paper, we assume existence of an impact model that satisfies the condition $|\dot{q}_+| = |\mathcal{D}(q, \dot{q}_-)| \leq \varepsilon |\dot{q}_-|$, where $0 < \varepsilon < 1$. The last $n-1$ equations of (10) can be used to compute the postimpact velocity $\dot{r}_+$ from the preimpact velocity $\dot{r}_-$, $\dot{q}_-$ and the impact model. Assuming Newton's impact model the postimpact velocity is given by

$$\dot{r}_+ = \dot{r}_- + (1 + \varepsilon)\dot{q}_- A^{-1}(q,r)b(q,r). \tag{11}$$

Also, the force impulse, f_{qI}, can be computed using the first equation of (10) and (11). The transformed dynamics of the system away from the surface can be rewritten as

$$\begin{bmatrix} \ddot{q} \\ \ddot{r} \end{bmatrix} + \begin{bmatrix} n_q(q, \dot{q}, r, \dot{r}) \\ N_r(q, \dot{q}, r, \dot{r}) \end{bmatrix} = \begin{bmatrix} v_q \\ v_r \end{bmatrix} \tag{12}$$

where

$$\begin{bmatrix} n_q(q, \dot{q}, r, \dot{r}) \\ N_r(q, \dot{q}, r, \dot{r}) \end{bmatrix} = \begin{bmatrix} a(q,r) & b^T(q,r) \\ b(q,r) & A(q,r) \end{bmatrix}^{-1} \begin{bmatrix} c(q, \dot{q}, r, \dot{r}) \\ C(q, \dot{q}, r, \dot{r}) \end{bmatrix}$$

and

$$\begin{bmatrix} v_q \\ v_r \end{bmatrix} = \begin{bmatrix} a(q,r) & b^T(q,r) \\ b(q,r) & A(q,r) \end{bmatrix}^{-1} \begin{bmatrix} u_q \\ u_r \end{bmatrix}.$$

The dynamics of the system away from the surface, (12), together with the generic impact model of (9) describe the dynamics of the nonsmooth system in the transformed coordinates. Controller design and closed-loop stability for stabilization of the system onto the surface is given in the following section.

III. CONTROLLER DESIGN AND STABILITY

The control objective is as follows. Given a nonzero initial condition, i.e., $q(0) = 0$ and $\dot{q}(0) \neq 0$, we want to design a stable control algorithm for constrained nonsmooth dynamic equations (8) and (9), for regulation of $q(t)$ and $\dot{q}(t)$ to zero and $r(t)$ to r_d, where r_d is the desired position of $r(t)$. For ease of expressing the controllers and the closed-loop error equations the following errors are defined: $e_{vq} = \dot{q} + \lambda_q q$, $e_r = r(t) - r_d$, and $e_{vr} = \dot{e}_r + \lambda_r e_r$, where e_{vq} and e_{vr} are the velocity reference errors in the variables q and r, respectively, and λ_q and λ_r are positive gains. Before designing the control algorithms, the following nonsmooth positive–definite Lyapunov function candidates are defined:

$$V_q(e_{vq}) = |e_{vq}| \tag{13}$$

$$V_r(e_{vr}) = \|e_{vr}\|_1 := \sum_{i=1}^{n-1} |e_{vri}| \tag{14}$$

where e_{vri} is the ith element of the vector e_{vr}. Wherever $\nabla V_q(e_{vq})$ and $\nabla V_r(e_{vr})$ exist (more precise details will be given), consider the following control algorithms:

$$\begin{aligned} v_q =& n_q(q, r, \dot{q}, \dot{r}) - \lambda_q a(q, r)\dot{q} - k_q \nabla V_q(e_{vq}) - \alpha_q q e_{vq} \\ & - \alpha_q \frac{\partial P_q(q)}{\partial q} \end{aligned} \tag{15}$$

$$v_r = N_r(q, r, \dot{q}, \dot{r}) - \lambda_q A(q, r)\dot{e}_r - k_r \nabla V_r(e_{vr}) \tag{16}$$

where k_q, k_r, and α_q are positive constants, $P_q = (1/2)q^2$, and ∇V represents the gradient of the function V. Notice that the control law (15) contains a nonlinear term and a potential force field term. The closed-loop error equations are

$$\dot{e}_{vq} = - k_q \nabla V_q(e_{vq}) - \alpha_q q e_{vq} - \alpha_q \frac{\partial P_q(q)}{\partial q} \text{ a.e.} \tag{17}$$

$$\dot{e}_{vr} = - k_r \nabla V_r(e_{vr}) \text{ a.e.} \tag{18}$$

The right-hand-side of the closed-loop error equations is nonsmooth. Before we prove stability of the solutions of the closed-loop error equations, we state some results from nonsmooth Lyapunov theory that will be used. Specifically, the solution concept for differential equations with discontinuous right-hand sides proposed by Filippov [11] is given, followed by a definition of Clarke's generalized gradient [12] and some results from [13] on computation of a time derivative of a nonsmooth Lyapunov function. Consider a system given by

$$\dot{z} = f(z) \tag{19}$$

where $z \in \mathbb{R}^n$ and $f(z)$ is a general discontinuous function. The following definition gives the solution of (19) in the sense of Filippov.

Definition 1 ([11]): An absolutely continuous vector function $z(t)\colon [t_0, t_1] \mapsto \Re^n$ is said to be a solution of (19) in the sense of Filippov if for almost all $t \in [t_0, t_1]$

$$\dot{z} \in \bigcap_{\delta > 0} \bigcap_{N} \overline{\text{co}} f(B(z(t), \delta) - N) =: K[f](z) \tag{20}$$

where $\overline{\text{co}}$ denotes the convex hull, $B(z(t), \delta)$ is a ball of radius δ centered at $z(t)$ and the intersection is taken over all sets N of measure zero.

The following definition gives Clarke's generalized gradient of a locally Lipschitz function.

Definition 2 ([12]): For a locally Lipschitz function $V\colon \Re^n \to \Re$ define the generalized gradient by

$$\partial V(z) = \overline{\text{co}} \{ lim \nabla V(z_i) | z_i \to z, z_i \notin \Omega_V \} \tag{21}$$

where Ω_V is the set of zero measure, where the gradient of V is not defined.

Now $e_{vq}(t)$ and $e_{vr}(t)$ are solutions of the closed-loop error equations (17) and (18), respectively, in the sense of Filippov, if

$$\dot{e}_{vq} \in - k_q K[\nabla V_q](e_{vq}) - \alpha_q q e_{vq} - \alpha_q \frac{\partial P_q(q)}{\partial q} \tag{22}$$

$$\dot{e}_{vr} \in - k_r K[\nabla V_r](e_{vr}) \tag{23}$$

where $K[\cdot](\cdot)$ is the set as defined in definition 1. Further, using the calculus for K (see [10, Th. 1]), $K[\nabla V](z) = \partial V(z)$, where $\partial V(z)$ is the Clarke's generalized gradient. Also, in what follows, we will assume existence and uniqueness of Filippov solution of the closed-loop error equations. Before we prove stability of the closed-loop solutions, we invoke the following two results from literature that will be used to prove finite time stability. The first is theorem 2.2 of [13], which defines $(d/dt)V(z)$ as an element of a set constructed from ∂V and $K[f](z)$. The second is [10, Th. 2], which is the Lyapunov theorem generalized to nonsmooth systems.

Theorem 1 ([13]): Let $z(t)$ be a Filippov solution to $\dot{z} = f(z)$ on an interval containing t and $V\colon \Re^n \to \Re$ be a Lipschitz and a regular function. Then $V(z)$ is absolutely continuous, $(d/dt)V(z)$ exists almost everywhere, and

$$\frac{d}{dt}V(z) \in^{\text{a.e.}} \dot{\tilde{V}}(z) \tag{24}$$

where

$$\dot{\tilde{V}}(z) := \bigcap_{\xi \in \partial V(z)} \xi^T K[f](z). \tag{25}$$

Theorem 2 ([10]): If 1) $V\colon \Re^n \to \Re$, $V(0) = 0$, and $V(z) > 0$, $\forall z \neq 0$ and 2) $z\colon \Re \to \Re^n$ and $V(z(t))$ is absolutely continuous on $[t_0, \infty)$ with

$$\frac{d}{dt}[V(z(t)] < -\epsilon \text{ a.e. on } \{t | z(t) \neq 0\} \tag{26}$$

where $\epsilon > 0$, then $z(t)$ converges to zero in finite time.

The proofs of both the theorems can be found in the respective references that are cited. The following theorem gives the stability of the solutions of the closed-loop error equations obtained with the proposed controller.

Theorem 3: For the differential equations describing the motion of the constrained Euler–Lagrange system, using the control algorithms given by (15) and (16), stable regulation of the system onto the constraint surface is achieved in finite time, i.e., q, $\dot{q}$, and e_r converge to zero in finite time.

Proof: Construct the following composite Lyapunov function candidate,

$$V(q, e_{vq}, e_{vr}) = \gamma V_q(e_{vq}) + V_r(e_{vr}) + \alpha_q P_q \tag{27}$$

where γ is a positive constant. Then

$$\begin{aligned} \dot{\tilde{V}}(q, e_{vq}, e_{vr}) =& \bigcap_{\xi_q \in \partial V_q} \gamma \left(-k_q \xi_q \partial V_q(e_{vq}) - \alpha_q q \xi_q e_{vq} - \alpha_q q \xi_q \right) \\ & - \bigcap_{\xi_r \in \partial V_r} k_r \xi_r^T \partial V_r(e_{vr}) \\ & - \alpha_q \lambda_q q^2 + \alpha_q q e_{vq} \end{aligned} \tag{28}$$

where we have used (22) and (23) and $K[\nabla V](z) = \partial V(z)$. Choose $\xi_q = \arg\min\{\|\nu_q\|\colon \nu_q \in \partial V_q\}$ and $\xi_r = \arg\min\{\|\nu_r\|\colon \nu_r \in \partial V_r\}$, then the convexity of the sets ∂V_q and ∂V_r gives

$$\begin{aligned} \dot{\tilde{V}}(q, e_{vq}, e_{vr}) \leq & -\gamma k_q \xi_q^2 - k_r \xi_r^T \xi_r - \alpha_q \lambda_q q^2 + \alpha_q \gamma |q| |\xi_q| \\ & -\alpha_q q(\gamma |e_{vq}| \xi_q^2 - e_{vq}). \end{aligned} \tag{29}$$

Notice that $\partial V_q = \mathrm{SGN}(e_{vq})$, where $\mathrm{SGN}(\cdot)$[1] denotes the set valued sign function of $(\cdot)$. Since V_q is convex, the set $\partial V_q(e_{vq}) \cap (-1, 1) = \emptyset$, for all nonzero e_{vq} (see [12] for details). Thus, $\xi_q^2 = 1$ for all nonzero e_{vq}. Similarly, since V_r is convex, the set $\partial V_r(e_{vr}) \cap (-1, 1)^{n-1} = \emptyset$, for all nonzero e_{vr}. Thus, $\xi_r^2 \geq 1$ for all nonzero e_{vr}. Also, since $q \geq 0$, and if $\gamma \geq 1$, then the last term in (29) is always positive, i.e., $\alpha_c q \left(\gamma |e_{vq}| \xi_q^2 - e_{vq}\right) \geq 0$. Therefore

$$\dot{\tilde{V}}(q, e_{vq}, e_{vr}) \leq -k_r \xi_r^T \xi_r - \gamma k_q \xi_q^2 - \alpha_q \lambda_q q^2 + \alpha_q \gamma |q| |\xi_q|. \quad (30)$$

Let $k_q = k_{qc} + \gamma$ and $\lambda_q = \gamma_c + (\alpha_q/4)$. Then

$$\dot{\tilde{V}}(q, e_{vq}, e_{vr}) \leq -k_r \xi_r^T \xi_r - \gamma k_{qc} \xi_q^2 - \alpha_q \gamma_c q^2 - (\gamma \xi_q - \frac{\alpha_q}{2} q)^2. \quad (31)$$

Since $\xi_r^T \xi_r \geq 1$ and $\xi_q^2 = 1$ for nonzero e_{vr} and e_{vq}, respectively, we obtain

$$\dot{\tilde{V}}(q, e_{vq}, e_{vr}) \leq -k_r - \gamma k_{qc}. \quad (32)$$

Hence, we can conclude from the generalized Lyapunov theorem (Theorem 2) that e_{vq}, e_{vr} and q converge to zero in finite time. To show that the Lyapunov function given by (27) is decreasing after every impact, we consider the post and preimpact Lyapunov functions given by

$$\begin{aligned} V_+ &= \gamma |\dot{q}_+ + \lambda_q q| + \|\dot{r}_+ + \lambda_r e_r\|_1 + \alpha_q P_q \quad (33) \\ V_- &= \gamma |\dot{q}_- + \lambda_q q| + \|\dot{r}_- + \lambda_r e_r\|_1 + \alpha_q P_q. \quad (34) \end{aligned}$$

Notice that we have used the fact that during impact position remains the same and the force impulse is in the normal direction. Also, notice that during impact the system is on the constraint surface, which means that $q = 0$ and $e_r = 0$. Using (9) and (11) in the above equations and simplifying, we obtain

$$\begin{aligned} V_+ - V_- \leq& \gamma |\dot{q}_+| - \gamma |\dot{q}_-| \\ &+ \left\| \dot{r}_- + (1+\varepsilon) \dot{q}_- A^{-1}(q, r) b(q, r) \right\|_1 - \|\dot{r}_-\|_1 \\ \leq& -\gamma (1-\varepsilon) |\dot{q}_-| + (1+\varepsilon) |\dot{q}_-| \left\| A^{-1}(q, r) b(q, r) \right\|. \end{aligned}$$

Since the inertia matrix is bounded from above and below, we can upper bound the expression $\|A^{-1}(q, r) b(q, r)\|$, say by η_b. Hence

$$V_+ - V_- \leq -\gamma(1-\varepsilon)|\dot{q}_-| + \eta_b (1+\varepsilon) |\dot{q}_-|. \quad (35)$$

Choosing $\gamma \geq \eta_b (1+\varepsilon)/(1-\varepsilon)$, we obtain $V_+ - V_- \leq 0$. The last term in the control law (15), and hence in (17), acts as a force field that directs the system toward the constraint surface. The quantity $P_q(q)$ can be thought of as a potential energy term for the coordinate q in the transformed system. □

IV. CONCLUSION

In this note, we have described the dynamics of the contact problem for constrained Euler–Lagrange systems by a set of nonsmooth differential equations. A new stable discontinuous control algorithm is designed to regulate the system onto the constraint surface in finite time. Our approach is unique in the sense that the control algorithm does not explicitly depend on the impact model and also the controller includes a potential term which assures return of the system onto the surface in the event of loss of contact due to unknown disturbances. Further, the nonsmooth model for the constrained dynamics and subsequent nonsmooth analysis depicts the natural behavior of the contact problem for Euler–Lagrange systems.

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An H_∞ Design Approach for Neural Net-Based Control Schemes

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***Abstract*—This note presents an H_∞ design approach for a neural net-based control scheme. In this scheme, a class of nonlinear systems is approximated by two multilayer perceptrons. The neural networks are piecewisely interpolated to generate a linear differential inclusion model. Based on this model, a state feedback control law is designed. The H_∞ control is specified to eliminate the effect of approximation errors and external disturbances to achieve desired performance. It is shown that finding the permissible control gain matrices can be transformed to a standard linear matrix inequality (LMI) problem and solved using the convex optimization method.**

***Index Terms*—H_∞ control, linear matrix inequality (LMI), neural network, robustness, stability.**

I. INTRODUCTION

Multilayer neural networks possess a number of interesting properties, such as the universal approximation capability and the possibility for on- and offline learning, which motivates their use for control applications [5]. In spite of reported successful neural control application, and that even for neural information storage application, energy function studies are mainly used for proof of convergence to desired values, there are not many stability analysis in neural control [3], [4], [8], [13]. Recently, stability conditions for a multilayer neural system, which is regarded as a linear differential inclusion (LDI) system, have been derived in [6], [14]. However, they had not analyzed the stability of neural control systems by considering modeling errors resulting from approximation of a plant with neural networks. For more about neural networks in related to the linear and nonlinear control theory see [11].

Regarding H_∞ control by neural networks, to the best of our knowledge, only a few results are published [11], [12]. In this note, the nonlinear state-space models are parameterized by multilayer perceptrons. An LDI state-space representation for a class of multilayer neural networks is established. Based on this representation, a linear state feedback control is considered. Design objectives regarding H_∞ regulation performance are discussed and the corresponding control laws are solved. The control design equations are characterized in the form of a set of linear matrix inequalities (LMIs) which allow for the application of convex optimization algorithms to be possible.

Throughout this note, denote the Euclidean norm $\|x\| = \sqrt{x^T x}$, the weighted Euclidean norm $\|x\|_Q = \sqrt{x^T Q x}$ where Q denotes the weighting matrix; $x\colon [0, \infty) \longmapsto \Re^n$ belongs to the space $L_2^n[0, \infty)$ with the norm $\|x(t)\|_{L_2} = \sqrt{\int_0^\infty x^T(t)x(t)dt}$, if $\|x(t)\|_{L_2} < \infty$.

II. NEURAL NET-BASED DESCRIPTION

Consider a nonlinear plant model described as follows:

$$\dot{x} = Ax + Bu + f(x) + g(u) + d \tag{1}$$

where

A, B	known constant matrices;
$x \in \Re^n$	state vector;
$u \in \Re^m$	input control vector;
$y \in \Re^z$	output vector;
$f(\cdot)\Re^n \longmapsto R^n$	continuous nonlinear mapping with $f(0) = 0$ but not assumed *a prior* known;
$g(\cdot)\Re^m \longmapsto R^m$	continuous nonlinear mapping with $g(0) = 0$ but not assumed *a prior* known;
$d \in L_2^n[0, \infty)$	uncertain external disturbances or unmodeled dynamics.

Parametrizations by neural nets will now be proposed for each of the model structures $f(x)$ and $g(u)$. Those make sense because any continuous nonlinear function can be approximated arbitrarily well on a compact interval by a multilayer neural network [5]. For more issues regarding model complexity and overfitting, etc., one is referred to [1]. Let the L-layered perceptrons $NN_x(x, W_1, W_2, \ldots, W_L)$ and $NN_u(u, V_1, V_2, \ldots, V_L)$, where $W_i\ (i = 1, \ldots, L) \in \Re^{n_i^x \times n_{i-1}^x}$ and $V_i\ (i = 1, \ldots, L) \in \Re^{n_i^u \times n_{i-1}^u}$ denote the weight matrices from the $(i-1)$th layer to the ith layer, be trained to approximate the uncertain terms $f(x)$ and $g(u)$, respectively. The I/O representations are formulated as

$$\begin{aligned} &NN_x(x, W_1, W_2, \ldots, W_L) \\ &\quad = \Psi_L[W_L \ldots \Psi_2[W_2 \Psi_1[W_1 x]] \ldots] \\ &NN_u(u, V_1, V_2, \ldots, V_L) = \tilde{\Psi}_L[V_L \ldots \tilde{\Psi}_2[V_2 \tilde{\Psi}_1[V_1 u]] \ldots] \end{aligned} \tag{2}$$

where the activation function vector $\Psi[\cdot]\colon \Re^n \mapsto \Re^n$ is defined as $\Psi[\nu] \triangleq [\,\psi_1(\nu_1) \quad \cdots \quad \psi_n(\nu_n)\,]^T$, $\tilde{\Psi}[\cdot]$ is defined similarly. Assume that all activation functions $\psi(\nu)$ are differentiable and satisfy i) $\psi(0) = 0$, ii) $\psi(\nu) \in [\lambda, -\lambda]$, $\lambda > 0, \forall\, t \geq 0, \forall\, \nu \in \Re$. The important characterization of the class of multilayer perceptrons is that all activation functions ψ associated with the hidden layers are the bipolar sigmoid type symmetric to the origin

$$F_s \triangleq \left\{ \psi(\cdot)\colon R \mapsto R \middle| \psi(\nu) = \lambda \left(\frac{1 - e^{-\nu/q}}{1 + e^{-\nu/q}} \right),\ q, \lambda > 0 \right\}.$$

The activation functions associated with the output layer are the linear type:

$$F_\ell \triangleq \{\psi(\cdot)\colon R \mapsto R | \psi(\nu) = \lambda\nu, \lambda > 0\}.$$

Suppose in the following that all connecting weights have been determined via a learning algorithm, such as the dynamic backpropagation rule [5]. According to the multilayer neural network theory, for given accuracy $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ over the compact sets $S_1 \in \Re^n$ and $S_2 \in \Re^m$ there exist op-

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