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Bounds on the solution of the time-varying linear matrix differential equation $\dot{P}(t) = A^{\mathrm{H}}(t)P(t) + P(t)A(t) + Q(t)$

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We derive upper and lower bounds for the trace of the solution of the time-varying linear matrix differential equation $\dot{P}(t) = A^{\mathrm{H}}(t)P(t) + P(t)A(t) + Q(t)$. A practical numerical example is given to verify the bounds. The bounds obtained are useful since the considered equation is encountered in a number of applications in systems and control theory.

Keywords: matrix differential equation; bounds; time-varying.

1. Introduction

The following notations are used: $\mathrm{tr}(M)$, $\lambda_{\min}(M)$, $\lambda_{\max}(M)$, M^{T} and M^{H} denote the trace, the minimum eigenvalue, the maximum eigenvalue, the transpose and the complex conjugate transpose of the matrix M, respectively. $M > 0$ ($\geqslant 0$) denotes that the matrix M is Hermitian positive definite (Hermitian positive semi-definite). $\mathbb{R}$ denotes the field of real numbers and $\mathbb{C}$ denotes the field of complex numbers. All matrices are assumed to be in $\mathbb{C}^{n\times n}$, unless otherwise specified. $\mu_M(M) \triangleq \lambda_{\max}((M + M^{\mathrm{H}})/2)$ and $\mu_m(M) \triangleq \lambda_{\min}((M + M^{\mathrm{H}})/2)$.

It is well known that the linear matrix differential equation of the following form

$$\dot{P}(t) = A^{\mathrm{H}}(t)P(t) + P(t)A(t) + Q(t) \tag{1}$$

plays an important role in systems, control and optimization (Brockett, 1970; Jodar & Ponsoda, 1995; Choi, 2003; Callier & Desoer, 1991; Gajic & Quershi, 1995). A number of applications of (1) and its special cases can be found in systems and control theory. In particular, consider the following time-varying linear system:

$$\dot{x}(t) = A_s(t)x(t) + B_s(t)u(t), \tag{2a}$$

$$y(t) = C_s(t)x(t) + D_s(t)u(t), \tag{2b}$$

where $A_s(\cdot)$, $B_s(\cdot)$, $C_s(\cdot)$, $D_s(\cdot)$ are piecewise continuous on $\mathbb{R}_+$ and the state space is $\mathbb{C}^n$. We use the convention given in Callier & Desoer (1991). The controllability and observability grammians, respectively, of the system (2) are given by

$$W_{\mathrm{c}}(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, \tau)B_s(\tau)B_s^{\mathrm{H}}(\tau)\Phi^{\mathrm{H}}(t_0, \tau)\mathrm{d}\tau, \tag{3}$$

$$W_{\mathrm{o}}(t_0, t_1) = \int_{t_0}^{t_1} \Phi^{\mathrm{H}}(\tau, t_0)C_s^{\mathrm{H}}(\tau)C_s(\tau)\Phi(\tau, t_0)\mathrm{d}\tau, \tag{4}$$

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where $\Phi(\cdot,\cdot)$ is the state transition matrix. The controllability grammian is the solution of (1) with $P(t_0) = 0$, $A(t) = A_s^{\mathrm{H}}(t)$, $Q(t) = B_s(t)B_s^{\mathrm{H}}(t)$ and $t \in [t_0, t_1]$. The observability grammian is the solution of (1) with $P(t_1) = 0$, $A(t) = -A_s^{\mathrm{H}}(t)$, $Q(t) = -C_s^{\mathrm{H}}(t)C_s(t)$ and $t \in [t_0, t_1]$. It is important to find the bounds on the grammians without explicitly solving the matrix differential equation.

Another motivating example of the use of (1) can be found in the evaluation of the integral (Brockett, 1970):

$$\alpha = \int_{t_0}^{t_1} x^{\mathrm{T}}(\tau)Q(\tau)x(\tau)\mathrm{d}\tau, \tag{5}$$

where $x(t)$ satisfies the following first-order differential equation

$$\dot{x}(t) = A(t)x(t). \tag{6}$$

The trajectory of (6) can be described by

$$x(t) = \Phi(t, t_0)x(t_0), \tag{7}$$

where $\Phi(t, t_0)$ is the transition matrix of (6). Substituting (7) in (5) yields

$$\alpha = x^{\mathrm{T}}(t_0)\left(\int_{t_0}^{t_1} \Phi^{\mathrm{T}}(\tau, t_0)Q(\tau)\Phi(\tau, t_0)\,\mathrm{d}\tau\right)x(t_0) \tag{8}$$

$$:= x^{\mathrm{T}}(t_0)W(t_1, t_0)x(t_0), \tag{9}$$

where $W(t_1, t_0)$ is the solution of (1) with $P(t_0) = 0$.

Trace and eigenvalue bounds on the solution of the following matrix differential equation, also called the Lyapunov matrix differential equation, can be found in Mori *et al.* (1987) and Hmamed (1990):

$$\dot{P}(t) = A^{\mathrm{T}}P(t) + P(t)A + Q, \tag{10}$$

where $A \in \mathbb{R}^{n\times n}$, $Q = Q^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, $Q \geqslant 0$ and A is stable. Notice that (1) is a more general case of (10). Upper and lower bounds for the trace or eigenvalues of the solution to (1) have not been reported in the literature. We derive upper and lower bounds for the trace of the solution to (1) in this paper.

The paper is organized as follows. In Section 2, the solution to a more general form of (1) is given and trace bounds on the solution (1) are derived. Section 3 presents a numerical example. Conclusions are given in Section 4.

2. Bounds on the solution of $\dot{P}(t) = A^{\mathrm{H}}(t)P(t) + P(t)A(t) + Q(t)$

The solution of a more general form of the linear matrix differential equation

$$\dot{P}(t) = A_1(t)P(t) + P(t)A_2(t) + Q(t), \quad P(t_0) = P_0, \tag{11}$$

where $A_1(t) \in \mathbb{C}^{n\times n}$, $A_2(t) \in \mathbb{C}^{n\times n}$ and $Q(t) \in \mathbb{C}^{n\times n}$ are bounded continuous functions of t, is given by (Brockett, 1970, p. 59)

$$P(t) = \Phi_1(t, t_0)P_0\Phi_2^{\mathrm{H}}(t, t_0) + \int_{t_0}^{t} \Phi_1(t, \tau)Q(\tau)\Phi_2^{\mathrm{H}}(t, \tau)\mathrm{d}\tau, \tag{12}$$

where $\Phi_1(t, t_0)$ and $\Phi_2(t, t_0)$, respectively, are the state transition matrices of the systems

$$\dot{x}_1(t) = A_1(t)x_1(t), \quad t \geqslant t_0, \tag{13}$$

$$\dot{x}_2(t) = A_2^{\mathrm{H}}(t)x_2(t), \quad t \geqslant t_0. \tag{14}$$

Since the solution to (11), given by (12), is unique (Coddington & Levinson, 1955), we can obtain the bounds on the solution based on the explicit form of the solution. First, we introduce several technical lemmas which are required to derive the bounds.

LEMMA 1 Let $M = M^{\mathrm{H}} \geqslant 0$ and $N = N^{\mathrm{H}}$, then

$$\lambda_{\min}(N)\mathrm{tr}(M) \leqslant \mathrm{tr}(MN) \leqslant \lambda_{\max}(N)\mathrm{tr}(M). \tag{15}$$

Proof. Since N is a Hermitian matrix, by Schur triangularization theorem (Marcus & Minc, 1964, p. 69), there exists a unitary matrix U such that

$$D = UNU^{\mathrm{H}}, \tag{16}$$

where D is a diagonal matrix whose diagonal elements are the eigenvalues of N.

Hence, we have

$$\begin{aligned}\mathrm{tr}(MN) &= \mathrm{tr}(U\ M\ U^{\mathrm{H}}U\ N\ U^{\mathrm{H}})\\ &= \mathrm{tr}(U\ M\ U^{\mathrm{H}}D).\end{aligned}$$

Since $M \geqslant 0$, which in turn implies that $UM\ U^{\mathrm{H}} \geqslant 0$, all diagonal elements of $UM\ U^{\mathrm{H}}$ are nonnegative real numbers. Hence, we have

$$\begin{aligned}\lambda_{\min}(N)\mathrm{tr}(UM\ U^{\mathrm{H}}) &\leqslant \mathrm{tr}(U\ M\ U^{\mathrm{H}}D)\\ &\leqslant \lambda_{\max}(N)\mathrm{tr}(U\ M\ U^{\mathrm{H}}).\end{aligned} \tag{17}$$

Notice that $\mathrm{tr}(UM\ U^{\mathrm{H}}) = \mathrm{tr}(M)$. Equation (15) follows. □

The inequality (15) is well known for the case when M and N are real symmetric positive definite (Mori *et al.*, 1987; Marcus & Minc, 1964; Patel & Toda, 1978; Kwon *et al.*, 1985), and for the case where both M and N are real, symmetric and M is positive definite (Wang *et al.*, 1986). Lemma 1 shows that (15) holds for any Hermitian matrices M and N.

LEMMA 2 Let $\Phi(t, t_0)$ be the transition matrix of the linear time-varying system

$$\dot{x} = A(t)x(t), \quad t \geqslant t_0, \tag{18}$$

where $A(t) \in \mathbb{C}^{n\times n}$. Then, for any $X = X^{\mathrm{H}} \in \mathbb{C}^{n\times n}$, the following is true.

1. If $X \geqslant 0$, then

$$\begin{aligned}\mathrm{tr}(X)\mathrm{e}^{\int_\tau^t 2\mu_m(A(\xi))\,\mathrm{d}\xi} &\leqslant \mathrm{tr}(\Phi(t,\tau)X\Phi^{\mathrm{H}}(t,\tau))\\ &\leqslant \mathrm{tr}(X)\mathrm{e}^{\int_\tau^t 2\mu_M(A(\xi))\,\mathrm{d}\xi}, \quad \forall t \geqslant \tau \geqslant t_0.\end{aligned} \tag{19}$$

2. If $X \leqslant 0$, then

$$\operatorname{tr}(X)\mathrm{e}^{\int_\tau^t 2\mu_M(A(\xi))\,\mathrm{d}\xi} \leqslant \operatorname{tr}(\Phi(t,\tau)X\Phi^{\mathrm{H}}(t,\tau)) \leqslant \operatorname{tr}(X)\mathrm{e}^{\int_\tau^t 2\mu_m(A(\xi))\,\mathrm{d}\xi}, \quad \forall\, t \geqslant \tau \geqslant t_0. \tag{20}$$

Proof. We give the proof for the first case. Using the property of $\Phi(t,\tau)$,

$$\frac{\mathrm{d}}{\mathrm{dt}}\Phi(t,\tau) = A(t)\Phi(t,\tau),$$

and the trace properties, $\operatorname{tr}(MN) = \operatorname{tr}(NM)$ and $\operatorname{tr}(M+N) = \operatorname{tr}(M) + \operatorname{tr}(N)$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}(\operatorname{tr}(\Phi(t,\tau)X\Phi^{\mathrm{H}}(t,\tau))) &= \operatorname{tr}\left(\frac{\mathrm{d}}{\mathrm{dt}}(\Phi(t,\tau)X\Phi^{\mathrm{H}}(t,\tau))\right)\\ &= \operatorname{tr}(A(t)\Phi(t,\tau)X\Phi^{\mathrm{H}}(t,\tau) + \Phi(t,\tau)X\Phi^{\mathrm{H}}(t,\tau)A^{\mathrm{H}}(t))\\ &= \operatorname{tr}((A(t) + A^{\mathrm{H}}(t))\Phi(t,\tau)X\Phi^{\mathrm{H}}(t,\tau)).\end{aligned}$$

Since $\Phi(t,t_0)X\Phi^{\mathrm{H}}(t,t_0) \geqslant 0$ and $(A(t) + A^{\mathrm{H}}(t))$ is a Hermitian matrix, we can apply Lemma 1 to the right-hand side of the above identity to get

$$2\mu_m(A(t))\operatorname{tr}(\Phi(t,\tau)X\Phi^{\mathrm{H}}(t,\tau)) \leqslant \frac{\mathrm{d}}{\mathrm{dt}}\operatorname{tr}(\Phi(t,\tau)X\Phi^{\mathrm{H}}(t,\tau)) \leqslant 2\mu_M(A(t))\operatorname{tr}(\Phi(t,\tau)X\Phi^{\mathrm{H}}(t,\tau)). \tag{21}$$

Notice that $\operatorname{tr}(\Phi(\tau,\tau)X\Phi^{\mathrm{H}}(\tau,\tau)) = \operatorname{tr}(X)$, and $\mu_M(A(t))$ and $\mu_m(A(t))$ are continuous functions. Solving the first-order scalar differential inequality (21) gives rise to (19).

For the second case, (20) can be obtained in a similar way as for the first case by considering the fact that $\Phi(t,\tau)(-X)\Phi^{\mathrm{H}}(t,\tau) \geqslant 0$. □

REMARK 1 Lemma 2 provides important inequalities which will be used in the derivation of bounds of the solution of (1). A similar known result in previous literature (Coppel, 1965) is given by the following inequality

$$\lambda_{\max}\left(\mathrm{e}^{At}\,\mathrm{e}^{A^{\mathrm{T}}t}\right) \leqslant \mathrm{e}^{2\mu_M(A)t}, \tag{22}$$

where $A \in \mathbb{R}^{n\times n}$. This inequality is used in Mori *et al.* (1987) to derive the upper and lower bounds on the solution to the linear matrix differential equation (10).

Notice that, in Lemma 2, the matrix $A(t)$ can be time varying. Applying Lemma 2 to the time-invariant case, one has

$$\mathrm{e}^{2\mu_m(A)t} \leqslant \frac{1}{n}\operatorname{tr}\left(\mathrm{e}^{At}\,\mathrm{e}^{A^{\mathrm{T}}t}\right) \leqslant \mathrm{e}^{2\mu_M(A)t}, \tag{23}$$

which gives upper and lower trace bounds.

Based on Lemmas 1 and 2, the bounds on the solution of the linear matrix differential equation and its special cases can be derived.

THEOREM 1 Consider the following linear matrix differential equation

$$\begin{aligned}\dot{P}(t) &= A^{\mathrm{H}}(t)P(t) + P(t)A(t) + Q(t),\\ P(t_0) &= P_0 = P_0^{\mathrm{H}} \geqslant 0,\end{aligned} \tag{24}$$

where $A(t) \in \mathbb{C}^{n\times n}$, $Q(t) = Q^{\mathrm{H}}(t) \in \mathbb{C}^{n\times n}$ and $Q(t) \geqslant 0$ are continuous functions of t. The trace of the solution to (24) is bounded by

$$\operatorname{tr}(P(t)) \leqslant \operatorname{tr}(P_0)\mathrm{e}^{\int_{t_0}^{t} 2\mu_M(A(\xi))\,\mathrm{d}\xi} + \int_{t_0}^{t} \operatorname{tr}(Q(\tau))\mathrm{e}^{\int_{\tau}^{t} 2\mu_M(A(\xi))\,\mathrm{d}\xi}\,\mathrm{d}\tau, \tag{25a}$$

$$\operatorname{tr}(P(t)) \geqslant \operatorname{tr}(P_0)\mathrm{e}^{\int_{t_0}^{t} 2\mu_m(A(\xi))\,\mathrm{d}\xi} + \int_{t_0}^{t} \operatorname{tr}(Q(\tau))\mathrm{e}^{\int_{\tau}^{t} 2\mu_m(A(\xi))\,\mathrm{d}\xi}\,\mathrm{d}\tau, \tag{25b}$$

for all $t \geqslant t_0$.

Proof. The solution to (24) is given by

$$P(t) = \Phi(t, t_0)P_0\Phi^{\mathrm{H}}(t, t_0) + \int_{t_0}^{t} \Phi(t, \tau)Q(\tau)\Phi^{\mathrm{H}}(t, \tau)\mathrm{d}\tau, \tag{26}$$

where $\Phi(t, t_0)$ is the transition matrix of the linear time-varying system

$$\dot{x}(t) = A^{\mathrm{H}}(t)x(t).$$

Since all the eigenvalues of $P(t)$ are real, taking trace on both sides of the solution, we have

$$\operatorname{tr}(P(t)) = \operatorname{tr}(\Phi(t, t_0)P_0\Phi^{\mathrm{H}}(t, t_0)) + \int_{t_0}^{t} \operatorname{tr}(\Phi(t, \tau)Q(\tau)\Phi^{\mathrm{H}}(t, \tau))\mathrm{d}\tau. \tag{27}$$

Applying Lemma 2 to (27) and using

$$\mu_M(A(\xi)) = \mu_M(A^{\mathrm{H}}(\xi)),$$

$$\mu_m(A(\xi)) = \mu_m(A^{\mathrm{H}}(\xi))$$

results in (25). □

REMARK 2 For the linear matrix differential equation (24), if the initial value of $P(t)$ satisfies $P(t_0) = P_0 = P_0^{\mathrm{H}} \leqslant 0$ and $Q(t) = Q^{\mathrm{H}}(t) \geqslant 0$, the trace bounds for the solution $P(t)$ are given by

$$\operatorname{tr}(P(t)) \leqslant \operatorname{tr}(P_0)\mathrm{e}^{\int_{t_0}^{t} 2\mu_m(A(\xi))\,\mathrm{d}\xi} + \int_{t_0}^{t} \operatorname{tr}(Q(\tau))\mathrm{e}^{\int_{\tau}^{t} 2\mu_M(A(\xi))\,\mathrm{d}\xi}\,\mathrm{d}\tau, \tag{28a}$$

$$\operatorname{tr}(P(t)) \geqslant \operatorname{tr}(P_0)\mathrm{e}^{\int_{t_0}^{t} 2\mu_M(A(\xi))\,\mathrm{d}\xi} + \int_{t_0}^{t} \operatorname{tr}(Q(\tau))\mathrm{e}^{\int_{\tau}^{t} 2\mu_m(A(\xi))\,\mathrm{d}\xi}\,\mathrm{d}\tau, \tag{28b}$$

for all $t \geqslant t_0$. Inequalities given by (28) are obtained by applying the inequality (20) to (27). Also, if $P(t_0) = P_0 = P_0^{\mathrm{H}} \geqslant 0$ and $Q(t) = Q^{\mathrm{H}}(t) \leqslant 0$, then

$$\operatorname{tr}(P(t)) \leqslant \operatorname{tr}(P_0)\mathrm{e}^{\int_{t_0}^{t} 2\mu_M(A(\xi))\,\mathrm{d}\xi} + \int_{t_0}^{t} \operatorname{tr}(Q(\tau))\mathrm{e}^{\int_{\tau}^{t} 2\mu_m(A(\xi))\,\mathrm{d}\xi}\,\mathrm{d}\tau, \tag{29a}$$

$$\operatorname{tr}(P(t)) \geqslant \operatorname{tr}(P_0)\mathrm{e}^{\int_{t_0}^{t} 2\mu_m(A(\xi))\,\mathrm{d}\xi} + \int_{t_0}^{t} \operatorname{tr}(Q(\tau))\mathrm{e}^{\int_{\tau}^{t} 2\mu_M(A(\xi))\,\mathrm{d}\xi}\,\mathrm{d}\tau. \tag{29b}$$

REMARK 3 Consider a time-invariant case of the linear matrix differential equation (24) given by

$$\dot{P}(t) = A^{\mathrm{T}} P(t) + P(t)A + Q, \quad P(0) = P_0 = P_0^{\mathrm{T}} > 0, \tag{30}$$

where $A \in \mathbb{R}^{n\times n}$, $Q = BB^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ and A is a stable matrix. The upper and lower bounds for the trace of the solution of (30) were investigated in Mori *et al.* (1987); the bounds were given by

$$\mathrm{tr}(P(t)) \leqslant \left(\mathrm{tr}(P_0) + \frac{\mathrm{tr}(Q)}{2\mu_M(A)}\right)\mathrm{e}^{2\mu_M(A)t} - \frac{\mathrm{tr}(Q)}{2\mu_M(A)}, \tag{31a}$$

$$\mathrm{tr}(P(t)) \geqslant \left(\mathrm{tr}(P_0) - \frac{\mathrm{tr}(Q)}{2\mu_M(-A)}\right)\mathrm{e}^{-2\mu_M(-A)t} + \frac{\mathrm{tr}(Q)}{2\mu_M(-A)}, \tag{31b}$$

for all $t \geqslant 0$.

Applying Theorem 1 to (30) with $t_0 = 0$, one has

$$\begin{aligned}\mathrm{tr}(P(t)) &\leqslant \mathrm{tr}(P_0)\mathrm{e}^{\int_0^t 2\mu_M(A)\,\mathrm{d}\xi} + \int_0^t \mathrm{tr}(Q)\mathrm{e}^{\int_\tau^t 2\mu_M(A)\,\mathrm{d}\xi}\,\mathrm{d}\tau \\ &= \mathrm{tr}(P_0)\mathrm{e}^{2\mu_M(A)t} + \int_0^t \mathrm{tr}(Q)\mathrm{e}^{2\mu_M(A)(t-\tau)}\,\mathrm{d}\tau \\ &= \left(\mathrm{tr}(P_0) + \frac{\mathrm{tr}(Q)}{2\mu_M(A)}\right)\mathrm{e}^{2\mu_M(A)t} - \frac{\mathrm{tr}(Q)}{2\mu_M(A)}, \end{aligned} \tag{32a}$$

$$\begin{aligned}\mathrm{tr}(P(t)) &\geqslant \mathrm{tr}(P_0)\mathrm{e}^{\int_0^t 2\mu_m(A)\,\mathrm{d}\xi} + \int_0^t \mathrm{tr}(Q)\mathrm{e}^{\int_\tau^t 2\mu_m(A)\,\mathrm{d}\xi}\,\mathrm{d}\tau \\ &= \left(\mathrm{tr}(P_0) + \frac{\mathrm{tr}(Q)}{2\mu_m(A)}\right)\mathrm{e}^{2\mu_m(A)t} - \frac{\mathrm{tr}(Q)}{2\mu_m(A)}. \end{aligned} \tag{32b}$$

Considering the fact that

$$\mu_m(A) = -\mu_M(-A),$$

we can observe that (31) and (32) are identical. We can also recover the bounds on the steady-state solution to (30) given in Mori *et al.* (1987).

If the solution to (26) as t approaches to infinity exists, (26) evaluated at $t = \infty$ gives the solution to the following Lyapunov matrix equation

$$A^{\mathrm{T}} P + PA + Q = 0. \tag{33}$$

When A is a stable matrix, $\mu_m(A) < 0$, the lower trace bound of the solution to (33) can be obtained from (31b) directly by replacing t with ∞. However, the upper trace bound of the steady-state solution may not be ascertained by directly applying (31a) because the right-hand side of (31a) may go to infinity, which is not a meaningful upper bound. When $\mu_M(A) < 0$, a finite upper bound can be obtained. Hence, the bounds on the solution of (33) are given by

$$-\frac{\mathrm{tr}(Q)}{2\mu_m(A)} \leqslant \mathrm{tr}(P) \leqslant -\frac{\mathrm{tr}(Q)}{2\mu_M(A)} \tag{34}$$

provided that $\mu_m(A) < 0$ and $\mu_M(A) < 0$.

Equation (34) illustrates the relationship between the traces of three matrices, A, P and Q, when (33) is satisfied. An important quantity related to the matrices P and Q for the control problem is the 'condition number' which is defined as $\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}$. It is usually desired to increase the condition number by changing the matrix A. In the following, we give an upper bound for the condition number.

Consider the fact that

$$\lambda_{\min}(Q) \leqslant \frac{\mathrm{tr}(Q)}{n}, \quad \lambda_{\max}(P) \geqslant \frac{\mathrm{tr}(P)}{n},$$

one has the following from (34)

$$\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} \leqslant \frac{\mathrm{tr}(Q)}{\mathrm{tr}(P)} \leqslant -2\mu_M(A). \tag{35}$$

From (35), we can notice that the condition number for the Lyapunov matrix equation (33) is bounded by the maximum eigenvalue of $A + A^{\mathrm{T}}$.

3. Example

Practical applications of linear time-varying systems can be found in a number of areas of engineering such as electrical circuits, structures subject to time-varying loading, helicopter rotor blades and many flight control applications (Balas & Lee, 1997; Lee & Choi). Consider the following linear time-varying system (Balas & Lee, 1997):

$$\dot{x}(t) = \begin{bmatrix} -\omega + \omega a \cos^2(\omega t) & \omega - \omega a \cos(\omega t)\sin(\omega t) \\ -\omega - \omega a \cos(\omega t)\ \sin(\omega t) & -\omega + \omega a \sin^2(\omega t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$\triangleq A_s(t)x(t) + B_s(t)u(t), \tag{36a}$$

$$y(t) = [1 \quad 0] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\triangleq C_s(t)x(t), \tag{36b}$$

where $a = 1/2, \omega = 2\pi$. The state transition matrix of (36) is given by

$$\Phi(t,0) = \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix} \begin{bmatrix} \mathrm{e}^{-\omega t/2} & 0 \\ 0 & \mathrm{e}^{-\omega t} \end{bmatrix}. \tag{37}$$

The controllability grammian of the system (36) is given by

$$W_{\mathrm{c}}(t) = \int_0^t \Phi(\tau,0)Q(\tau)\Phi^{\mathrm{H}}(\tau,0)\,\mathrm{d}\tau$$

$$= \int_0^t \begin{bmatrix} \mathrm{e}^{-2\omega\tau}\sin^2(\omega t) & \mathrm{e}^{-2\omega\tau}\cos(\omega t)\sin(\omega t) \\ \mathrm{e}^{-2\omega\tau}\cos(\omega t)\sin(\omega t) & \mathrm{e}^{-2\omega\tau}\cos^2(\omega t) \end{bmatrix} \mathrm{d}\tau. \tag{38}$$

Hence,

$$\operatorname{tr}(W_c(t)) = \int_0^t e^{-2\omega\tau} \sin^2(\omega t)\, d\tau + \int_0^t e^{-2\omega\tau} \cos^2(\omega t)\, d\tau$$
$$= \frac{1}{2\omega}(1 - e^{-2\omega t}). \tag{39}$$

Recall that the controllability grammian is the solution of (1) with $P(0) = 0$, $A(t) = A_s^{H}(t)$, $Q(t) = B_s(t)B_s^{H}(t)$ and $t \in [0, t_1]$. To verify if the bounds given by (25a) and (25b) are correct, we compute them and compare with (39). Notice that

$$\mu_m(A) = -\omega, \quad \mu_M(A) = -\omega/2.$$

Applying the bounds from Theorem 1 with $P(0) = 0$ results in the lower bound and upper bound for $P(t)$, which is equivalent to $W_c(t)$, as

$$\operatorname{tr}(P(t)) \geqslant \int_0^t e^{-2\omega(t-\tau)}\, d\tau = \frac{1}{2\omega}(1 - e^{-2\omega t}), \tag{40a}$$

$$\operatorname{tr}(P(t)) \leqslant \int_0^t e^{-\omega(t-\tau)}\, d\tau = \frac{1}{\omega}(1 - e^{-\omega t}). \tag{40b}$$

Comparing (39) and (40), it can be observed that (40) gives trace bounds for the controllability grammian for the example system (36).

4. Conclusion

We derived upper and lower bounds for the trace of the solution to the time-varying linear matrix differential equation. Previous work (Mori *et al.*, 1987; Hmamed, 1990) gave bounds for the time-invariant linear matrix differential equation. The results of this work can be applied to a broader class of linear systems, i.e. for both time-invariant and time-varying systems. A practical numerical example is given to verify the usefulness of the bounds.

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