Adaptive Estimation of Time-Varying Parameters in Linearly Parametrized Systems

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Adaptive estimation of time-varying parameters in linearly parametrized systems is considered. The estimation time is divided into small intervals; in each interval the time-varying parameter is approximated by a time polynomial with unknown coefficients. A condition for resetting of the parameter estimate at the beginning of each interval is derived; the condition guarantees that the estimate of the time-varying parameter is continuous and also allows for the coefficients of the polynomial to be different in various time intervals. A modified version of the least-squares algorithm is provided to estimate the time-varying parameters. Stability of the proposed algorithm is shown and discussed. Simulation results on an example are given to validate the proposed method. [DOI: 10.1115/1.2234488]

1 Introduction

It is evident from a study of the literature that an important motivation for designing adaptive controllers is in dealing with time-varying parameters. Even though the research in identification and control of time-varying systems has been active during the past two decades, the adaptive estimation of time-varying parameters in linearly parametrized systems is still an open problem. Most adaptive estimation algorithms, such as the least-squares and the gradient algorithms and a number of variations of them, have nice stability and convergence properties in the ideal case when the parameters are constant [1,2]. But these algorithms fail to retain most of their properties when the parameters are time varying.

Adaptive control of a class of slowly time-varying discrete-time systems is considered in [3]; it is shown that the traditional gradient algorithm designed for the estimation of the constant parameters can maintain stability when the plant parameters are slowly time varying. In [4], time-varying linear systems in linear parametrized form with modeling error is considered for adaptive control design; a gradient algorithm with projection is used to estimate the time-varying parameters; it is shown that the parameter estimation error is bounded under the assumption that the parameter variations are uniformly small and the modeling errors are bounded by a small exponentially weighted sum of plant inputs and outputs. Model reference adaptive control with slowly time-varying plants can be found in [5]. A number of results in adaptive control of linear-time varying plants can be found in [6]. In [7], a comparative survey with respect to performance and robustness between recursive and direct least-squares estimation algorithms is presented; a nonrecursive algorithm that improves robustness to bounded disturbances for the case of slowly time-varying parameters is given.

In [8], it is shown via simulation results that applying local regression in traditional least-squares with a forgetting factor algorithm can reduce the estimation error in the mean-square sense for systems with slowly time-varying parameters. The adaptive control of discrete-time systems with time-varying parameters can be found in [9,10]; a polynomial approximation of the time-varying parameters in a discrete sense is used in the parameter estimation algorithms. Nonparameteric regression techniques to various statistical problems, using local polynomial modeling, are discussed extensively in [11]. In [13], an adaptive controller is developed for time-varying mechanical systems based on polynomial approximations of time-varying parameters and disturbances; experimental results of the adaptive controller on a planar robot are given to verify the proposed adaptive controller.

Systems given by the linear parametric model, where the unknown parameters are time varying, are considered in this work. Since a wide class of systems can be represented by linear parameter models [2,6,11], extensive research can be found on this topic. In particular, one practical application that is the motivation for this research is in control of robots used in filling and pouring operations, where the payload is time varying. The dynamics of the robot in such situations can be linearly parametrized [12,13]?

A local polynomial approximation in a finite time interval is used to represent the unknown time-varying parameters. The coefficients of the polynomials are estimated locally instead of the unknown time-varying parameter. The accuracy of the approximation depends on the order of the polynomial and the width of the time interval, which can be chosen. The polynomial coefficients vary from one interval to the other, but within an interval they are constant. Thus, each time-varying parameter is approximated independently in each interval by a set of constant coefficients. Based on the approximation, a modification to a traditional least-squares algorithm with covariance resetting is provided for the linear time-varying parametric model. Stability of the modified algorithm is shown and discussed. Simulation results for the proposed algorithm on an example are presented.

The following are the contributions of the paper: (1) A local polynomial approximation model is proposed for linearly parametrized systems; a condition that for continuous resetting of the parameter estimate at the beginning of each interval is derived. (2) Based on the model developed, a modified least-squares algorithm with covariance resetting is proposed, and stability properties of the algorithm are given.

The rest of the paper is organized as follows. Section 2 gives the problem statement. A representation of the time-varying parameters via local polynomials is discussed in Sec. 3. The linear time-varying parametric model in terms of the local polynomial approximations is described in Sec. 4. The modified least-squares algorithm is discussed in Sec. 5. Section 6 gives simulation results for an example. Conclusions are given in Sec. 7.

2 Problem Statement

Consider the continuous-time system given by the parametric model

$$z(t) = \theta^{*\mathsf{T}}(t)\Phi(t) \tag{1}$$

where $\theta^{*\mathsf{T}}(t)=[\theta_0^*(t),\theta_1^*(t),\ldots,\theta_m^*(t)] \in \mathbb{R}^m$ is the unknown time-varying parameter vector, $\Phi(t)=[\Phi_1(t),\Phi_2(t),\ldots,\Phi_m(t)]^{\mathsf{T}} \in \mathbb{R}^m$ is the known signal vector, and $z(t) \in \mathbb{R}$ is the measured output. It is assumed that $\theta^*(t)$ belongs to the class of piecewise continuous m-times differentiable functions, that is,

$$\theta^*(t) \in \{\theta^{*(m)}(t) \in \mathcal{L}_\infty,\ m=1,\ldots,p\} \tag{2}$$

where $\theta^{*(m)}(t)$ denotes the mth time derivative of $\theta^*(t)$. The goal is to design a parameter adaptation law for tracking the time-varying

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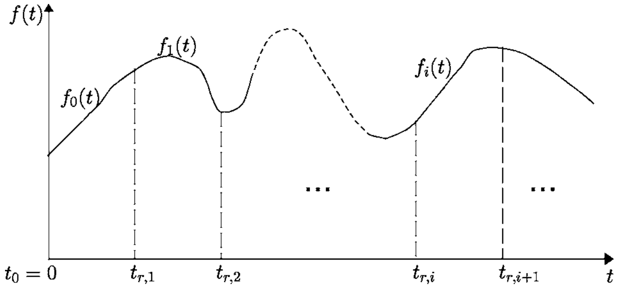

Fig. 1 Local polynomial approximation of continuous function

parameter $\theta^*(t)$ based on the signal vector $\Phi(t)$ and the measured output $z(t)$.

3 Representation of Time-Varying Parameters

To represent a time-varying parameter, consider the following result [14]:

LEMMA 1. *Let I be an open interval in $\mathbb{R}$, and f be a $(p+1)$-times continuously differentiable function of I into $\mathbb{R}$; then, for any pair of points t_0, t in I,*

$$f(t) = f(t_0) + \frac{(t-t_0)}{1!} f^{(1)}(t_0) + \cdots + \frac{(t-t_0)^p}{p!} f^{(p)}(t_0) + \int_{t_0}^{t} \frac{(t-\xi)^{p+1}}{(p+1)!} f^{(p+1)}(\xi) d\xi \tag{3}$$

where $f^{(i)}(\cdot)$ denotes the ith derivative of the function $f(\cdot)$.

If the function $f(t)$ is approximated by the first $p+1$ terms in (3), then the last term, $\delta := \int_{t_0}^{t} [(t-\xi)^{p+1}/(p+1)!] f^{(p+1)}(\xi) d\xi$, represents the error due to the approximation. If the $(p+1)$th derivative of $f(t)$ is bounded, then the approximation error can be made small by choosing a small interval $(t-t_0)$.

As a result of Lemma 1, the time-varying function can be approximated locally at t_0 as a polynomial of time with constant coefficients, that is,

$$f(t) = a_0(t_0) + a_1(t_0)(t-t_0) + \cdots + a_p(t_0)(t-t_0)^p, t \in [t_0, t_0+T)$$

$$:= \sum_{i=0}^{p} a_i(t_0)(t-t_0)^i \tag{4}$$

where $a_i(t_0) = (1/i!) f^{(i)}(t_0)$, $i=0,\ldots,p$, $f^{(i)}(t_0)$ is the ith time derivative evaluated at $t=t_0$, and T is the window length that can be chosen. Assuming the window is sufficiently small, the last term of (3) is negligible; that is, $\delta := \int_{t_0}^{t} [(t-\xi)^{p+1}/(p+1)!] f^{(p+1)}(\xi) d\xi$ is negligible. Suppose that the $(p+1)$th derivative of $f(t)$ is bounded, that is, $\sup_t \| f^{(p+1)}(t) \| \leq c_p$, then δ can be bounded by

$$|\delta| \leq \frac{c_p (t-t_0)^{p+1}}{(p+1)!}. \tag{5}$$

Therefore, it is possible to use (4) to approximate $f(t)$ closely by choosing either a higher order polynomial, that is, p large, or a small interval T such that $t - t_0 \leq T$, or both. If we choose t_0 as a nondecreasing sequence of time instants with each difference between adjacent t_0 not more than T, in other words, the partition time into segments with the length of each segment not larger than T, then the time-varying function $f(t)$ can be approximated by a number of polynomials of time locally at each t_0 with constant coefficients a_i; Fig. 1 illustrates the idea, where $f_i(t), i=0,1,\ldots$, locally represents the function $f(t)$ by a polynomial in the ith window. In general, the coefficients a_i between two intervals are different.

The function $f(t)$ can also be approximated locally at $t_r \neq t_0$ by

$$f(t) = a_0(t_r) + a_1(t_r)(t-t_r) + \cdots + a_p(t_r)(t-t_r)^p \tag{6}$$

$$:= \sum_{i=0}^{p} a_i(t_r)(t-t_r)^i \tag{7}$$

where $a_i(t_r) = (1/i!) f^{(i)}(t_r)$. To express each $a_j(t_r)$ in terms of $a_i(t_0), i=0,\ldots,p$, evaluate the jth derivative of (6) and (4) at $t = t_r$; notice that one can do this under the assumption that $t_r - t_0 \leq T$. The jth derivative of (4) and (6) are:

$$f^{(j)}(t) = \sum_{i=0}^{p} a_i(t_0) \frac{i!}{(i-j)!} (t-t_0)^{i-j}, \tag{8}$$

$$f^{(j)}(t) = \sum_{i=0}^{p} a_i(t_r) \frac{i!}{(i-j)!} (t-t_r)^{i-j}. \tag{9}$$

Evaluating (8) at $t=t_r$, we obtain

$$f^{j}(t_r) = \sum_{i=0}^{p} a_i(t_0) \frac{i!}{(i-j)!} (t_r - t_0)^{i-j} = \sum_{i=0}^{p} a_i(t_r) \frac{i!}{(i-j)!} (t_r - t_r)^{i-j}.$$

Similarly, evaluating (9) at $t=t_r$,

$$f^{j}(t_r) = \sum_{i=0}^{p} a_i(t_r) \frac{i!}{(i-j)!} (t_r - t_r)^{i-j} = a_j(t_r).$$

Combining the above two equations results in

$$a_j(t_r) = \sum_{i=0}^{p} a_i(t_0) \frac{i!}{j!(i-j)!} (t-t_r)^{i-j}. \tag{10}$$

Therefore, the relationship between $a_j(t_r), j=0,\ldots,p$, and $a_i(t_0), i=0,\ldots,p$, is given by

$$\begin{bmatrix} a_0(t_r) \\ a_1(t_r) \\ \vdots \\ a_p(t_r) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & t_r - t_0 & \cdots & (t_r - t_0)^p \\ 0 & 1 & \cdots & p(t_r - t_0)^{p-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}}_{A(t_r, t_0)} \begin{bmatrix} a_0(t_0) \\ a_1(t_0) \\ \vdots \\ a_p(t_0) \end{bmatrix}. \tag{11}$$

4 Local Polynomial Approximation Model

Applying the local polynomial approximation to each element of the time-varying parameter vector $\theta^*(t)$ locally at t_0, that is,

$$\theta_i^*(t) = \theta_{i0} + \theta_{i1}(t-t_0) + \cdots + \theta_{ip}(t-t_0)^p := \theta_i^{\mathsf{T}}(t_0) L(t, t_0) \tag{12}$$

where $\theta_i(t_0) := [\theta_{i0}(t_0), \theta_{i1}(t_0), \ldots, \theta_{ip}(t_0)]^{\mathsf{T}}$ is the unknown constant vector and $L(t,t_0) := [1, (t-t_0), \ldots, (t-t_0)^p]^{\mathsf{T}}$ is a column vector. Notice that $\theta_i^*(t)$ is the original time-varying parameter that is being approximated by the time polynomial with coefficients $\theta_{i0}, \theta_{i1}, \ldots, \theta_{ip}$. If $t_{r,i}$ is defined as the time instant at which the ith window of the local polynomial approximation begins, then t_0 is given by the sequence $t_0 = \{t_{r,i}\}$ with $i=0,1,\ldots$, and $t_{r,i+1} - t_{r,i} = T$. In the following $t_{r,i}$ is referred to as the resetting time, which is the beginning of the ith window of the local polynomial approximation. Notice that $\theta_i(t_0)$ is constant only within each interval $[t_{r,i}, t_{r,i+1})$ and, in general, differs from one interval from another for a time-varying parameter. The parameter vectors at two adjacent resetting times, $\theta_i(t_{r,i})$ and $\theta_i(t_{r,i+1})$, are related by (11), that is, $\theta_i(t_{r,i+1}) = A(t_{r,i+1}, t_{r,i}) \theta_i(t_{r,i})$. The polynomial order p can be chosen for different $\theta_i^*(t)$ based on some *a priori* knowledge; for convenience, p is chosen to be the same for all the time-varying parameters. Therefore, the original parameter vector $\theta^*(t)$ is related to the approximation polynomial coefficient vector $\theta(t_0)$ by

$$\theta^*(t) = \begin{bmatrix} L^{\mathsf{T}}(t,t_0) & & & \\ & L^{\mathsf{T}}(t,t_0) & & \\ & & \ddots & \\ & & & L^{\mathsf{T}}(t,t_0) \end{bmatrix} \theta(t_0) := \Lambda(t,t_0)\theta(t_0) \tag{13}$$

where $\Lambda(t,t_0)$ is an $m \times m(p+1)$ matrix. Equation (1) and the resetting times can be written as

$$\begin{gathered} z(t) = \theta^{*\mathsf{T}}(t)\phi(t) = \theta^{\mathsf{T}}(t_0)\Psi(t,t_0), \\ t_0 = \{t_{r,i}\}, \quad i = 0,1,2,\ldots, \end{gathered} \tag{14}$$

where $\Psi(t,t_0) = \Lambda^{\mathsf{T}}(t,t_0)\Phi(t)$. As $\theta(t_0)$ is now a piecewise constant vector, the problem of estimating the time-varying parameter in (1) can be transformed to that of estimating the constant parameter in (14) based on the observations within each interval $[t_0, t_0+T)$. Consequently, various estimation algorithms designed for estimating constant parameters may be employed with appropriate modifications.

By using (11), $\theta(t_{r,i+1})$ and $\theta(t_{r,i})$ are related by the following equation:

$$\begin{aligned} \theta(t_{r,i+1}) &= \begin{bmatrix} A(t_{r,i+1},t_{r,i}) & & & \\ & A(t_{r,i+1},t_{r,i}) & & \\ & & \ddots & \\ & & & A(t_{r,i+1},t_{r,i}) \end{bmatrix} \theta(t_{r,i}) \\ &:= B(t_{r,i+1},t_{r,i})\theta(t_{r,i}). \end{aligned} \tag{15}$$

Notice that $\theta(t_{r,i})$ is constant in the ith interval, that is, $\theta(\tau) = \theta(t_{r,i})$ for all $\tau \in [t_{r,i}, t_{r,i+1}^-]$. Equation (15) will form the basis for resetting the initial value of the estimate at the beginning of each interval, and Eq. (14) will be used to identify the constant coefficients of the polynomial in each time interval. In the next section, the least squares and the gradient algorithms are modified to estimate the time-varying parameter vector by introducing a resetting scheme at the beginning of each interval; the resetting scheme ensures that the estimate of the time-varying parameter vector, $\hat{\theta}^*(t)$, is continuous, consistent with the assumptions on the true time-varying parameters. Stability properties of each identification algorithm with the proposed resetting scheme is shown and discussed.

Remark 1. Extensive work has been done in estimation of parameters in autoregressive models with exogenous inputs (ARX). ARX models with time-varying coefficients, whose variation is limited to a set of predefined functions, could be used for the estimation of time-varying coefficients. A problem that could arise with such an approach is that the estimates of the coefficients will be influenced by the choice of the set of predefined functions rather than the data itself. Local polynomial modeling is more datacentric and depends on the window of the interval chosen in which the data is being collected (see [11] for discussions related to this aspect).

5 Modified Least-Squares Algorithm

Based on the local polynomial approximation described in the previous section, a modified version of the least-squares with covariance resetting algorithm is given for identification of time-varying parameters. The least-squares algorithm with covariance resetting has been widely used for estimating an unknown constant parameter vector, β, for the following model:

$$y(t) = \beta^{\mathsf{T}}\phi(t) \tag{16}$$

where $\phi(t)$ is a known signal vector. The estimate of β, $\hat{\beta}$, is given by minimizing the following integral cost function

$$J(\beta) = \frac{1}{2}\int_0^t [y(\tau) - \beta^{\mathsf{T}}(t)\phi(\tau)]^2\, d\tau, \tag{17}$$

that is, $\hat{\beta} = \arg\min_\beta J(\beta)$. The minimization of (17) with covariance resetting results in the following estimation algorithm [2]:

$$\begin{gathered} \dot{\hat{\beta}} = P\phi(y - \hat{\beta}^T\phi) \\ \dot{P} = -P\phi\phi^{\mathsf{T}}P, \quad P(t_{\mathrm{cr}}^+) = \rho_0 I \end{gathered} \tag{18}$$

where t_{cr} is the time at which $\lambda_{\min}[P(t)] \leqslant \rho_1$, $\lambda_{\min}(\cdot)$ denotes the smallest eigenvalue of a matrix, I is the identity matrix, and $\rho_0 > \rho_1 > 0$ are some design scalars.

In the following, a modified version of the above algorithm where, in addition to the covariance resetting, the initial value of the estimate is reset at the beginning of each time window of the local polynomial approximation. For the time-varying model given by (14), choose the cost function as follows:

$$J(\hat{\theta}) = \frac{1}{2}\int_0^t \epsilon^2(t,\tau)m^2(\tau)d\tau \tag{19}$$

where

$$\epsilon(t,\tau) = \frac{z(\tau) - \hat{\theta}^{\mathsf{T}}(t)\Psi(\tau)}{m^2(\tau)}$$

is the normalized estimation error, $m^2(\tau) = 1 + n_s^2$, and $n_s^2 = \Psi^{\mathsf{T}}(\tau)\Psi(\tau)$. The adaptation law is chosen as follows:

$$\begin{gathered} \dot{\hat{\theta}}(t) = P(t)\epsilon(t,t)\Psi(t) \\ \dot{P}(t) = -\frac{P\Psi\Psi^{\mathsf{T}}P}{m^2}, \quad P(t_0) = \rho_0 I \end{gathered} \tag{20}$$

where $\hat{\theta}(t)$ is the estimate of $\theta(t_0)$. Further, the covariance matrix is reset as follows:

$$P(t) = \rho_0 I, \quad \text{if } \lambda_{\min}[P(t)] \leqslant \rho_1. \tag{21}$$

Equation (21) ensures that the covariance matrix does not get too close to singularity, that is, the covariance matrix is reset within each time window if its minimum eigenvalue becomes less than ρ_1. At the beginning of each window the initial value of the estimate is reset according to the following equation:

$$\hat{\theta}(t_{r,i+1}) = B(t_{r,i+1},t_{r,i})\hat{\theta}(t_{r,i+1}^-). \tag{22}$$

The motivation for resetting the initial value at the beginning of each interval by (22) is the following: Since the true parameter $\theta_i^*(t)$ is continuous, the estimate, $\hat{\theta}_i^*(t)$, should also be continuous; (22) guarantees this at the resetting points $t_{r,i}$. The following shows the continuity of $\hat{\theta}^*(t)$ at the resetting point. Just before resetting for the $(i+1)$-th interval, using (13) for the estimate, we obtain

$$\hat{\theta}^*(t_{r,i+1}) = \Lambda(t_{r,i+1}^-, t_{r,i})\hat{\theta}(t_{r,i+1}^-) \tag{23}$$

At the resetting point, again using (13) for the estimate with $t_0 = t_{r,i+1}$,

$$\hat{\theta}^*(t_{r,i+1}) = \Lambda(t_{r,i+1},t_{r,i+1})\hat{\theta}(t_{r,i+1}) = \Lambda(t_{r,i+1},t_{r,i+1})B(t_{r,i+1},t_{r,i})\hat{\theta}(t_{r,i+1}^-), \tag{24}$$

where (22) has been used to obtain the second equality. Notice that

$$\Lambda(t_{r,i+1},t_{r,i+1}) = \mathrm{diag}(e_1^{\mathsf{T}}, e_1^{\mathsf{T}}, \ldots, e_1^{\mathsf{T}})$$

where $e_1^{\mathsf{T}} = [1,0,\ldots,0]$. Also, from the definition of $B(t_{r,i+1},t_{r,i})$ given by (15), we obtain

$$\Lambda(t_{r,i+1},t_{r,i+1})B(t_{r,i+1},t_{r,i}) = \mathrm{diag}[e_1^{\mathsf{T}}A(t_{r,i+1},t_{r,i}), e_1^{\mathsf{T}}A(t_{r,i+1},t_{r,i}), \ldots, e_1^{\mathsf{T}}A(t_{r,i+1},t_{r,i})] = \Lambda(t_{r,i+1},t_{r,i}).$$

Therefore, (24) becomes

$$\hat{\theta}^*(t_{r,i+1}) = \Lambda(t_{r,i+1},t_{r,i})\hat{\theta}(t_{r,i+1}^-) \tag{25}$$

From (23) and (25), since $\Lambda(t,t_0)$ is a continuous function of t, it can be seen that $\hat{\theta}^*(t_{r,i+1}^-)=\hat{\theta}^*(t_{r,i+1})$.

The following theorem gives the stability of the modified least-squares algorithm with covariance resetting for the time-varying model.

THEOREM 5.1: *The least-squares algorithm given by* (20), *together with the covariance resetting given by* (21) *and the resetting of the estimate*, $\hat{\theta}(t)$, *at the beginning of each interval, given by* (22), *has the following properties:*

(i) $\epsilon, \epsilon n_s, \hat{\theta}, \dot{\hat{\theta}} \in \mathcal{L}_\infty$.

(ii) $\epsilon, \epsilon n_s, \dot{\hat{\theta}} \in \mathcal{L}_2$.

(iii) If $n_s, \Psi \in \mathcal{L}_\infty$ *and* Ψ *satisfies the following persistence of the excitation (PE) condition:*

$$\alpha_1 T_0 I \geq \int_t^{t+T_0} \Psi(\tau)\Psi^{\mathsf{T}}(\tau)d\tau \geq \alpha_0 T_0 I, \qquad \forall t \geq 0 \quad \text{and} \quad T_0 < T, \tag{26}$$

for some $0<\alpha_0\leq\alpha_1$, *then* $\hat{\theta}(t)$ *converges exponentially to* $\theta(t_0)$.

(iv) The estimate of $\theta^*(t)$, $\hat{\theta}^*(t)$, *is continuous and bounded. Furthermore, if* Ψ *satisfies the PE condition given in (iii), the estimation error* $\tilde{\theta}^*(t)$ *exponentially converges to zero within each time interval.*

Proof: Consider the Lyapunov function candidate

$$V[\tilde{\theta}(t)] = \frac{\tilde{\theta}^{\mathsf{T}}(t)P^{-1}(t)\tilde{\theta}(t)}{2}. \tag{27}$$

It can be shown that, within an interval $(t\in[t_0,t_0+T))$, the derivative of the Lyapunov function candidate satisfies:

$$\dot{V}(t) = -\frac{\epsilon^2 m^2}{2} \leq 0. \tag{28}$$

Thus, one can can arrive at (*i*), (*ii*), and (*iii*) of the theorem as given in [2]. Notice that in (iii) there is an additional constraint in the PE condition (26), that is, $T_0<T$. This is necessary for the coefficients of the local polynomial approximation to converge to their true values within any interval. The following gives the proof of (*iv*).

The estimate of $\theta^*(t)$ for $t\in[t_0,t_0+T)$, $\hat{\theta}^*(t)$, is given by

$$\hat{\theta}^*(t) = \Lambda(t,t_0)\hat{\theta}(t). \tag{29}$$

Hence,

$$\|\hat{\theta}^*(t)\| \leq \|\Lambda(t,t_0)\|\,\|\hat{\theta}(t)\| \leq \sqrt{\lambda_{\max}[\Lambda^{\mathsf{T}}(t,t_0)\Lambda(t,t_0)]}\|\hat{\theta}(t)\| \leq \kappa(T)\|\hat{\theta}(t)\| \tag{30}$$

where $\lambda_{\max}[\Lambda^{\mathsf{T}}(t,t_0)\Lambda(t,t_0)]=1+(t-t_0)^2+\cdots+(t-t_0)^{2p}$ is the maximum eigenvalue of $\Lambda^{\mathsf{T}}(t,t_0)\Lambda(t,t_0)$ and $\kappa(T)=\sqrt{1+T^2+\cdots+T^{2p}}$. The boundedness of $\hat{\theta}^*(t)$ follows from the fact that $\hat{\theta}(t)$ is bounded. Also, taking the time derivative of (29), we obtain

$$\dot{\hat{\theta}}^*(t) = \dot{\Lambda}(t,t_0)\hat{\theta}(t) + \Lambda(t,t_0)\dot{\hat{\theta}}(t). \tag{31}$$

$\dot{\hat{\theta}}^*(t)$ is bounded within each time interval because $\dot{\Lambda}(t,t_0)$ and $\Lambda(t,t_0)$ are bounded within each time interval, and $\hat{\theta}(t)$ and $\dot{\hat{\theta}}(t)$ are bounded [from (*i*)]. Hence, $\hat{\theta}^*(t)$ is continuous within each time interval. Recall that the continuity of $\hat{\theta}^*(t)$ at each resetting point is guaranteed by the resetting of the estimate at the beginning of each time interval according to (22). Therefore, it follows that $\hat{\theta}^*(t)$ is uniformly continuous.

Subtracting (13) from (29) yields

$$\tilde{\theta}^*(t) = \Lambda(t,t_0)\tilde{\theta}(t). \tag{32}$$

Therefore, the estimation error, $\tilde{\theta}^*(t)$, is bounded by

$$\|\tilde{\theta}^*(t)\| \leq \kappa(T)\|\tilde{\theta}(t)\|. \tag{33}$$

Recall that, from (*iii*), $\tilde{\theta}(t)$ exponentially converges to zero, which implies that $\tilde{\theta}^*(t)$ exponentially converges to zero within each interval. □

Rate of convergence: In the following an estimate of the rate of convergence of the parameters is derived. The least-squares algorithm, (20), satisfies [2]:

$$\tilde{\theta}(t) = P(t)P^{-1}(t_0)\tilde{\theta}(t_0), \quad t\in[t_0,t_0+T) \tag{34}$$

and

$$P(t) \leq [(t-t_0-T_0)\alpha_0]^{-1}\bar{m}I, \quad \forall\, t\geq t_0+T_0 \tag{35}$$

where $\bar{m}=\sup_t m^2(t)$. So, the worst case bound of $\tilde{\theta}(t)$ is given by

$$\|\tilde{\theta}(t)\| \leq \|P(t)\|\,\|P^{-1}(t_0)\|\,\|\tilde{\theta}(t_0)\| \leq [\rho_0(t-t_0-T_0)\alpha_0]^{-1}\bar{m}\|\tilde{\theta}(t_0)\|. \tag{36}$$

At the end of the ith interval, that is, $t=iT+T^-$, we have

$$\|\tilde{\theta}(iT+T^-)\| \leq [\rho_0(T-T_0)\alpha_0]^{-1}\bar{m}\|\tilde{\theta}(iT)\|. \tag{37}$$

From (32), we have

$$\|\tilde{\theta}^*(iT+T)\| \leq \kappa(T)\|\tilde{\theta}(iT+T^-)\| \leq \kappa(T)[\rho_0(T-T_0)\alpha_0]^{-1}\bar{m}\|\tilde{\theta}(iT)\|. \tag{38}$$

Notice that, from (37) and (38), faster convergence of the estimate of the time-varying parameter vector, $\hat{\theta}^*(t)$, and the vector of coefficients of the polynomial, $\hat{\theta}(t)$, within a time interval depends on how small T_0 is with respect to T. Further, it also depends on the persistency of excitation level of the signal vector $\Psi(t)$ (α_0) and ρ_0.

6 Simulations

Consider the following first-order system given in [7]:

$$z(t) = \theta_1^*(t)u_f(t) + \theta_2^*(t)z_f(t) + n(t) = \theta^{*\mathsf{T}}(t)\phi(t) + n(t) \tag{39}$$

where $z(t)$ is the output of the plant, $n(t)$ is the noise, $\theta^{*\mathsf{T}}(t)=[\theta_1^*(t),\theta_2^*(t)]^{\mathsf{T}}$, $\phi(t)=[u_f(t),z_f(t)]^{\mathsf{T}}$, and the filtered input and output signals, $u_f(t)$ and $z_f(t)$, are given by

$$\dot{u}_f(t) = -300u_f(t) + 300u(t), \quad \dot{z}_f(t) = -300z_f(t) + 300z(t), \tag{40}$$

where $u(t)$ is the input. The input $u(t)$ is chosen to be a random signal with zero mean and a variance of 0.01. In the simulation, $\theta_1^*(t)$ is approximated by a sixth-order polynomial of time, and $\theta_2^*(t)$ is approximated by a first-order polynomial of time. The following values are used in the simulations: $T=0.1$ s, $\rho_0=2400$, and $\rho_1=0.005$. The following three sets of simulations are shown for different sets of time-varying parameters (Table 1).

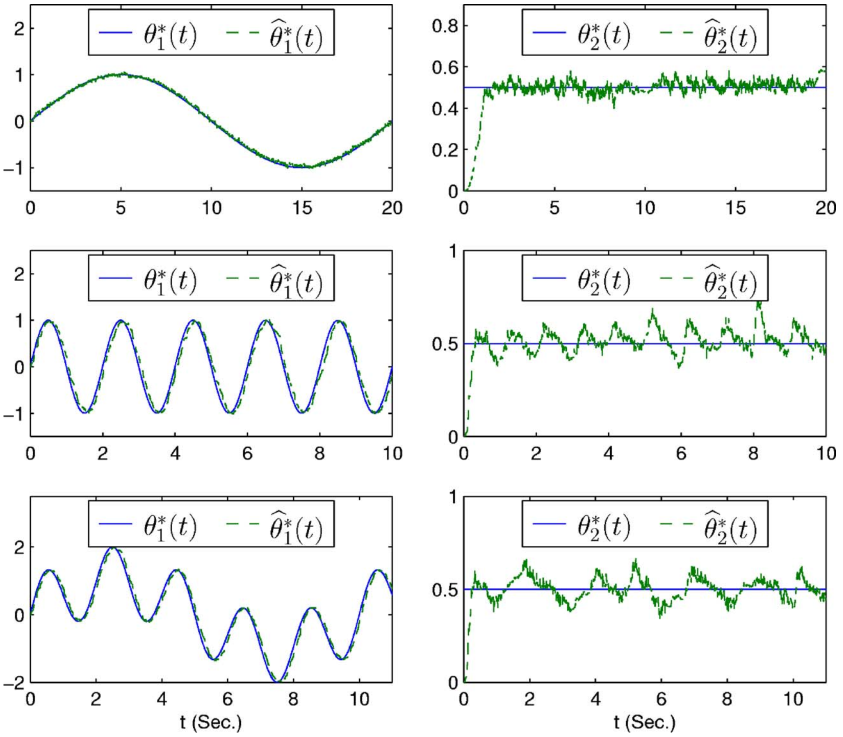

Fig. 2 Simulation results

Figure 2 shows the simulation results. Plots in the first column and the second column show the first parameter and its estimate, and second parameter and its estimate, respectively. Plots in rows 1, 2, and 3 show the simulation results corresponding to the parameter sets 1, 2, and 3, respectively, given in Table 1. Simulation results show that the estimate converges to a small region around the true value for the proposed modified least-squares algorithm.

7 Conclusions

We developed a modified version of the traditional least-squares algorithm for adaptive estimation of unknown time-varying parameters in linear parametric models. The time-varying parameters were approximated locally in small intervals of time by truncated Taylor series expansion in finite intervals of time. A strategy to reset the initial value of the parameter estimate at the beginning of each time interval was given; this assures that the parameter estimate is continuous at the resetting points. Stability and convergence properties of the proposed estimation algorithm was given. Simulation results conducted on an example verify the proposed algorithm. One particular feature of the method described is that the time-varying parameters are not assumed to be slowly time varying, because both the parameters and their time derivatives are estimated locally. Although the estimation algorithm is developed in the continuous-time domain, it can be extended to the discrete-time domain under the assumption of fast sampling.

Table 1 Simulation parameters

Set	$\theta_1^*(t)$	$\theta_2^*(t)$	$n(t)$
1	$\sin(\pi t/10)$	0.5	$N(0,0.01)$
2	$\sin(\pi t)$	0.5	$N(0,0.01)$
3	$\sin(\pi t)+\sin(\pi t/5)$	0.5	$N(0,0.01)$

Future research will focus on robustness of the proposed algorithm to modeling error in the linear parametrized model, sensitivity analysis of the algorithm with respect to noise, and experimentation of the algorithm on a practical system.

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