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A note on the necessary conditions for the algebraic Riccati equation

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We derive some useful and easily computable necessary conditions for the existence of a positive semi-definite solution to the algebraic Ricatti equation (ARE). A motivating example is given to highlight the usefulness of the conditions for controller and observer designs for nonlinear systems. Further, an upper bound on the trace of the solution to the ARE is also derived.

Keywords: algebraic Ricatti equations (ARE); nonlinear systems; trace bounds.

1. Introduction

The following notation is used. $\mathrm{tr}(M), \lambda(M), \lambda_{\min}(M), \lambda_{\max}(M)$, M^H, and $M^\top$ denote the trace, eigenvalues, the minimum eigenvalue, the maximum eigenvalue, the complex conjugate transpose, and the transpose of the matrix M, respectively. $M > 0$ ($\geqslant 0, < 0, \leqslant 0$) denotes that the matrix M is symmetric, positive definite (positive semi-definite, negative definite, negative semi-definite). $\mathrm{Re}(c)$ and $\bar{c}$ denote the real part and the complex conjugate, respectively, of the complex number c. $|a|$ denotes the absolute value of a. All the matrices are assumed to be real square matrices of dimension $n \times n$, unless explicitly indicated.

It is well known that the ARE of the following form plays an important role in systems and control:

$$A^\top P + PA + PRP + Q = 0 \tag{1}$$

where $R \geqslant 0$, $Q \geqslant 0$, A is the system matrix, and P is the solution to (1). Finding necessary and sufficient conditions for the existence of a positive semi-definite solution to the ARE is of considerable interest. Extensive work has been reported on this topic in the literature (Bittanti *et al.*, 1991; Lancaster & Rodman, 1995; Zhou *et al.*, 1996). Further, there has been a strong interest in determining the bounds on the solutions to the Lyapunov equations and the AREs which appear in the systems and control areas (Patel & Toda, 1978; Yasuda & Hirai, 1979; Wang *et al.*, 1986). In this note, some useful necessary conditions for the existence of a positive semi-definite solution to the ARE (1) are derived. The necessary conditions obtained are easily computable.

Notice that in (1) we consider the case where $R \geqslant 0$. In the case where $R = -BB^\top \leqslant 0$ and $Q = C^\top C \geqslant 0$, the ARE (1) becomes

$$A^\top P + PA - PBB^\top P + C^\top C = 0 \tag{2}$$

where $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$. The ARE (2) has been extensively studied (Kalman, 1961; Wonham, 1968; Kucera, 1972a,b; Kano, 1987). It is shown in Kucera (1972a,b) that the ARE (2) has a unique

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positive semi-definite solution P if and only if (C, A) is detectable and (A, B) is stabilizable. Further, in Kano (1987), it is shown that if (A, B) is stabilizable, then the ARE (2) has a unique positive definite solution if and only if $(C, -A)$ is detectable. However, thus far, necessary and sufficient conditions for the existence of a positive semi-definite solution to the ARE (1) have not been found.

In the following, we give an example of the controller design for Lipschitz nonlinear systems, where an ARE of the form given by (1) arises.

2. A motivating example

Consider the linear state-feedback controller for the following class of Lipschitz nonlinear systems:

$$\dot{x} = A_o x + Bu + \Phi(x, u), \tag{3}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the system state and input, respectively. We make the following assumptions on the system (3): (1) (A_o, B) is a controllable pair, (2) $x = 0$ is the equilibrium point of the system, (3) $\Phi(x, u)$ is globally Lipschitz with respect to the state x, uniformly in the control u with a known Lipschitz constant γ, such that

$$\|\Phi(x_1, u) - \Phi(x_2, u)\| \leqslant \gamma \|x_1 - x_2\| \tag{4}$$

for all $x_1, x_2 \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, and (4) $\Phi(x, u)$ is such that $\|\Phi(x, u)\| \leqslant \gamma\|x\|, \forall u \in \mathbb{R}^m$.

Consider the control law $u = -kx$ where K is the feedback gain matrix. With this control input, the closed-loop dynamics is

$$\dot{x} = (A_0 - BK)x + \Phi(x, u) \overset{\triangle}{=} Ax + \Phi(x, u). \tag{5}$$

To show stability, we consider the following Lyapunov function candidate:

$$V(x) = x^\top P x \tag{6}$$

where P is a symmetric, positive definite matrix. The time derivative of the Lyapunov function candidate along the trajectories of (5) is

$$\begin{aligned} \dot{V}(x) &= x^\top \Big(A^\top P + PA\Big)x + 2x^\top P\Phi(x, u) \\ &\leqslant x^\top \Big(A^\top P + PA\Big)x + 2\gamma\|Px\|\|x\| \\ &\leqslant x^\top \Big(A^\top P + PA + PP + \gamma^2 I\Big)x \end{aligned} \tag{7}$$

where the first inequality is a consequence of assumption (3) and the second inequality is obtained by completing squares on the term $2\gamma\|Px\|\|x\|$. For any $\eta_c > 0$, if

$$A^\top P + PA + PP + \gamma^2 I = -\eta I, \tag{8}$$

then $\dot{V}(x) \leqslant -\eta x^\top x$, and hence the control law $u = -Kx$ asymptotically stabilizes the system (3) if there exists a positive definite solution to (8). Now, the problem reduces to finding a positive definite solution to the ARE (8), which is in the form of (1). Similarly, the observer design problem for the system (3) also gives rise to solving an ARE of the form (1) (Pagilla & Zhu, 2004).

The following question naturally arises: Under what conditions does the ARE (1) not have a positive definite solution? The answer to the question can be found in the following theorem which gives necessary conditions to the existence of a positive semi-definite (definite) solution to the ARE (1).

3. Main result

THEOREM 1 Suppose that $P \geqslant 0$ is a solution to the ARE (1). It is necessary that the following be true:

$$\lambda_{\min}(R)\mathrm{tr}(Q) - n\lambda^2_{\min}(S) < 0, \tag{9a}$$

$$\lambda_{\min}(S) < 0, \tag{9b}$$

where $S = (A + A^\top)/2$.

Proof. Taking trace on both sides of the ARE (1) results in

$$\mathrm{tr}(A^\top P) + \mathrm{tr}(PA) + \mathrm{tr}(PRP) + \mathrm{tr}(Q) = 0. \tag{10}$$

Using the matrix trace property $\mathrm{tr}(MN) = \mathrm{tr}(NM)$, we obtain

$$\mathrm{tr}(A^\top P) + \mathrm{tr}(PA) = 2\mathrm{tr}(PS). \tag{11}$$

Now, consider the following inequalities (Wang *et al.*, 1986):

$$\lambda_{\min}(S)\mathrm{tr}(P) \leqslant \mathrm{tr}(PS) \leqslant \lambda_{\max}(S)\mathrm{tr}(P), \tag{12}$$

$$\lambda_{\min}(R)[\mathrm{tr}(P)]^2/n \leqslant \mathrm{tr}(PRP) \leqslant \lambda_{\max}(R)[\mathrm{tr}(P)]^2. \tag{13}$$

Let $x \overset{\triangle}{=} \mathrm{tr}(P)$. Using (11), (12) and (13) in (10) yields

$$2\lambda_{\min}(S)x + \frac{\lambda_{\min}(R)}{n}x^2 + \mathrm{tr}(Q) \leqslant 0. \tag{14}$$

Equation (14) is equivalent to

$$\left(x + n\frac{\lambda_{\min}(S)}{\lambda_{\min}(R)}\right)^2 + \frac{n}{\lambda_{\min}(R)^2}\left(\lambda_{\min}(R)\mathrm{tr}(Q) - n\lambda_{\min}(S)^2\right) \leqslant 0, \quad \text{if } \lambda_{\min}(R) > 0, \tag{15a}$$

$$2\lambda_{\min}(S)x + \mathrm{tr}(Q) \leqslant 0, \quad \text{if } \lambda_{\min}(R) = 0. \tag{15b}$$

From (15a), it is clear that the second term in the left-side of the inequality must be negative. Hence, (9a) must be true. From (15), the upper bound on x is given by

$$x \leqslant \frac{n}{\lambda_{\min}(R)}\left(-\lambda_{\min}(S) + \sqrt{\lambda_{\min}(S)^2 - \frac{\lambda_{\min}(R)\mathrm{tr}(Q)}{n}}\right), \quad \text{if } \lambda_{\min}(R) > 0, \tag{16a}$$

$$x \leqslant -\frac{\mathrm{tr}(Q)}{2\lambda_{\min}(S)}, \quad \text{if } \lambda_{\min}(R) = 0. \tag{16b}$$

From (16), it is clear that (9) must be true. □

REMARK 1 Inequality (16) gives the trace upper bound of the solution to the ARE (1).

LEMMA 1 If A is Hurwitz, then $\lambda_{\min}(S) < 0$.

If A is Hurwitz, then the necessary condition (9b) is satisfied. The proof of the lemma is quite straightforward and is given below. Let $v \neq 0$ be an eigenvector of A corresponding to an eigenvalue of A. Then

$$Av = \lambda(A)v. \tag{17}$$

Using (17), we can obtain

$$v^H(A + A^\top)v = (\lambda(A) + \bar{\lambda}(A))v^H v. \tag{18}$$

Because A is Hurwitz, the real part of any eigenvalue of A is negative. Hence

$$(\lambda(A) + \bar{\lambda}(A))v^H v = 2\mathrm{Re}(\lambda(A))v^H v < 0. \tag{19}$$

From (18), (19), and S being symmetric, one has

$$\lambda_{\min}(S)v^H v \leqslant v^H S v = \mathrm{Re}(\lambda(A))v^H v < 0. \tag{20}$$

Therefore, $\lambda_{\min}(S) < 0$.

REMARK 2 Consider the example given in Section 2 again. Since (A_o, B) is a controllable pair, K can be chosen such that A is Hurwitz. From Lemma 1, the condition (9b) is satisfied. However, the condition (9a) may not be satisfied because of the structure of matrices A and B, and the value of γ. For example, assume

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad K = \begin{bmatrix} k_1 & k_2 \end{bmatrix},$$

and the eigenvalues of A are -1 and -2. Then, we have

$$K = \begin{bmatrix} 2 & 3 \end{bmatrix}, \; A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix},$$

$$S = \begin{bmatrix} 0 & -1/2 \\ -1/2 & -3 \end{bmatrix}, \; \lambda_{\min}(S) = -(3 + \sqrt{10}\,)/2.$$

Obviously, inequality (9b) is satisfied. To assure (9a) to be true, the Lipschitz constant γ must satisfy

$$\gamma = \sqrt{\mathrm{tr}(Q)/2} \; < \sqrt{\lambda^2_{\min}(S)/\lambda_{\min}(R)} \; = (3 + \sqrt{10}\,)/2.$$

Example: Consider the following example:

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q = \varepsilon \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{21}$$

where $a_1, a_2 \in \mathbb{R}$ and $\varepsilon > 0$. Let $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$. For any a_1, a_2, one solution of the ARE is

$$\begin{cases} p_1 = -a_1 \pm \sqrt{a_1^2 - \varepsilon} \\ p_2 = 0 \\ p_3 = -a_2 \pm \sqrt{a_2^2 - \varepsilon} \end{cases} . \tag{22}$$

If $|a_1| = |a_2|$, in addition to (22), we also have other solutions given by

$$\begin{cases} p_1 = -a_1 \pm \sqrt{a_1^2 - \varepsilon - p_2^2} \\ p_3 = -a_2 \mp \sqrt{a_2^2 - \varepsilon - p_2^2} \end{cases} \tag{23}$$

and p_2 is arbitrary.

For (22), the necessary and sufficient conditions for P to be a symmetric positive definite matrix are

$$a_1 < 0, \ a_2 < 0, \ \varepsilon - a_1^2 < 0, \ \text{and } \varepsilon - a_2^2 < 0. \tag{24}$$

Similarly, the necessary and sufficient conditions for P to be a symmetric positive definite matrix, for the solutions given by (23) ($|a_1| = |a_2|$), are

$$a_1 < 0, \ a_2 < 0, \ \varepsilon - a_1^2 - p_2^2 < 0, \ \text{and } \varepsilon - a_2^2 - p_2^2 < 0. \tag{25}$$

The condition given by (9) for this example is equivalent to

$$\varepsilon - a_{\min}^2 < 0, \tag{26a}$$

$$a_{\min} < 0 \tag{26b}$$

where $a_{\min} = \min(a_1, a_2)$. For (24) and (25) to be true, it is necessary that (26) must be true.

4. Conclusion

In conclusion, we have derived some useful and easily computable necessary conditions for the existence of a positive semi-definite solution to a class of AREs which are important in systems and control. An upper bound on the trace of the solution is also obtained.

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